**ACCREDITATION SCHEME FOR LABORATORIES**

# Technical Guide 1

# Guidelines on the Evaluation and Expression of Measurement Uncertainty

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Third Edition, 18 March 2026

## ACKNOWLEDGEMENT

**FOREWORD**

ISO/IEC 17025: 1999: General requirements for the competence of testing and calibration laboratories, which replaces ISO/IEC Guide 25, requires all calibration and testing laboratories performing their own calibration to have a procedure to estimate the uncertainty of measurement for all calibrations and types of calibrations. SAC-SINGLAS has adopted this new standard as the criteria for accrediting competent testing and calibration laboratories.

In October 1998, a Working Group was convened to review and update the first edition of SINGLAS Technical Guide 1 "Guidelines on the evaluation and expression of the measurement uncertainty" published in July 1995, which was based on "Guide to the Expression of Uncertainty in Measurement, 1993(E)" or in short the GUM.

The second edition of the Technical Guide 1 was revised to align industry practice with the revised GUM (1995) and it contains more worked examples. These examples automatically supersede those in the first edition.

ISO/IEC Guide 25:1990: General Requirements for the Competence of Calibration and Testing Laboratories, required that "a statement of the estimated uncertainty of the calibration or test results" be reported in the calibration and test reports. In order to assist accredited calibration laboratories meet this requirement, in 1993, the SINGLAS Technical Committee on Calibration and Measurement decided that it was necessary to produce a simplified document for deriving measurement uncertainty.

# CONTENTS

# 1.0    INTRODUCTION

1.1    Increasingly, many have recognized that correct evaluation of measurement uncertainty is integral to managing quality and costs in measurements. Uncertainty analysis leads to better understanding of the relative importance of various influence quantities on measurements.

1.2    This publication gives recommendations for the treatment of uncertainty contributions and is intended to be applicable to most, if not all, of SAC-SINGLAS accredited measurement results. This publication does not define or recommend what the uncertainty contributions are, or should be, since these are dependent on the nature of the calibration and the measurement process. Worked examples on the treatment of uncertainty are included for the temperature, electrical, mechanical, dimensional and optical areas.

# 2.0    CLASSIFICATION OF COMPONENTS OF UNCERTAINTY

2.1    The objective of a measurement is to determine the value of the measurand that is, the value of the particular quantity to be measured.

2.2    In general, the result of a measurement is only an approximation or estimate of the value of the measurand and thus is complete only when accompanied by a statement of the uncertainty of that estimate.

2.3    Traditionally, an error in the measurement result is viewed as having two components, namely, a random component and a systematic component. Random errors presumably arise from unpredictable or stochastic temporal and spatial variations of influence quantities. Systematic error arises from a recognized effect of an influence quantity on a measurement result.

2.4    The uncertainty of the result of a measurement generally consists of several components which may be grouped into two types according to the method used to estimate their numerical values:

    Type A - those evaluated by statistical methods,
    Type B - those evaluated by other means.

2.5    There is not always a simple correspondence between the classification of uncertainty components into categories A and B and the commonly used classification of uncertainty components as "random" and "systematic". The nature of an uncertainty component is conditioned by the use of the corresponding quantity, that is, how that quantity appears in the mathematical model that describes the measurement process. When the corresponding quantity is used in a different way, a "random" component may become a "systematic" component and vice versa. Thus the terms "random uncertainty" and "systematic uncertainty" can be misleading when generally applied. An alternative nomenclature that might be used is

    "component of uncertainty arising from a random effect,"
    "component of uncertainty arising from a systematic effect,"

where a random effect is one that gives rise to a possible random error in the current measurement process and a systematic effect is one that gives rise to a possible systematic error in the current measurement process.  In principle, an uncertainty component arising from a systematic effect may in some cases be evaluated by method A while in the other cases by method B, as may be an uncertainty component arising from a random effect.


## 3.0      MODELLING THE MEASUREMENT

3.1      The output measurand $Y$ is usually determined from $N$ input quantities, ie. $X_1, X_2, ..., X_N$, through a function $f$:

$$Y = f(X_1, X_2, \cdots X_N) \qquad\qquad ...(1)$$

3.2      The input quantities $X_1, X_2, ..., X_N$ may themselves be viewed as measurands and may themselves depend on other quantities, including corrections and correction factors for systematic effects, thereby leading to a complicated functional relationship $f$.

3.3      The function $f$ contains every quantity that contribute a significant component of uncertainty to the measurement result - this includes all corrections and correction factors. If the function $f$ does not model the measurement to the extent set by the required accuracy of the measurement result, additional input quantities must be included in $f$ to eliminate the inadequacy.

3.4      An estimate of the measurand $Y$, denoted by $y$, is obtained from equation (1) using input estimates $x_1, x_2, ..., x_N$, for the values of the $N$ input quantities $X_1, X_2, ..., X_N$.  The output estimate $y$, which is the result of the measurement, is thus given by:

3.5      The estimated standard deviation associated with the output estimate $y$ is obtained by

$$y = f(x_1, x_2, \cdots x_N) \qquad\qquad ...(2)$$

appropriately combining the estimated standard deviation (termed as standard uncertainty and denoted by $u(x_i)$) of each input estimate $x_i$.

3.6      Each standard uncertainty $u(x_i)$ is obtained either from a Type A or Type B evaluation as described below. Type A evaluations of standard uncertainty components are founded on frequency distributions while Type B evaluations are founded on a priori distributions - in both cases these distributions are models that represent the current state of knowledge.

## 4.0    TYPE A EVALUATION OF STANDARD UNCERTAINTY

4.1    Type A evaluation of standard uncertainty is based on any valid statistical method in analysis of a series of observations.

4.2    A component of Type A evaluation of standard uncertainty arises from random effect. The Gaussian or Normal law of error forms the basis of the analytical study of random effects.

4.3    In most cases, the best available estimate of the expected value of a quantity $q$ that varies randomly, is the arithmetic mean $\bar{q}$. The arithmetic mean for $n$ independent observations is given by:

$$\bar{q} = \frac{1}{n} \sum_{k=1}^{n} q_k \qquad ...(3)$$

4.4    The experimental standard deviation $s(q_k)$ is used to estimate the distribution of $q$.

$$s(q_k) = \sqrt{\frac{1}{n-1} \sum_{k=1}^{n} \left( q_k - \bar{q} \right)^2} \qquad ...(4)$$

4.5    The experimental standard deviation of the mean $s(\bar{q})$ is used to estimate the spread of the distribution of the means.

$$s(\bar{q}) = \frac{s(q_k)}{\sqrt{n}} \qquad ...(5)$$

4.6    One important factor deciding the number of readings required in a measurement process is the permissible precision error of the parameter. Averaging a set of readings is a common technique for reducing the random error of the result. However, the benefit gets progressively less as the number is increased and it is usually not necessary to make more than ten measurements.[3]

4.7    For a quantity determined from $n$ independent repeated observations, the Type A standard uncertainty $u$, with degrees of freedom $v$ is :

$$u(x_i) = s(\bar{q}) \qquad ...(6)$$

$$v_i = n - 1 \qquad ...(7)$$

4.8    The degrees of freedom $v$ should always be given when Type A evaluation of uncertainty components are documented.

4.9    It is sometimes commercially impractical to investigate the Type A standard uncertainty for every measurement. Where a practical type of measurement is performed frequently, provided there are no change in the measurement system or procedure, it may be sufficient to estimate an uncertainty in terms of an investigation carried out earlier, and verified at discrete intervals of time to ensure that the measurement has not been degraded.

4.10   For a well-characterized measurement under statistical control, a pooled experimental standard deviation $s_p$, with degrees of freedom $\nu_p$, based on $M$ series of observations of the same variable may be available. The pooled experimental standard deviation is determined by:

$$s_p^2 = \frac{\sum_{i=1}^{M} \nu_i \, s_i^2}{\sum_{i=1}^{M} \nu_i} \qquad ...(8)$$

$$\nu_p = \sum_{i=1}^{M} \nu_i \qquad ...(9)$$

where $s_i$ is the experimental standard deviation from one series of $m_i$ independent repeated observations, and has degrees of freedom of $\nu_i = m_i - 1$.

4.11   If a measurand $\bar{q}$ of the same variable is determined from $n$ independent observations, the Type A standard uncertainty $u$ can be estimated by:

$$u(x_i) = \frac{s_p}{\sqrt{n}} \qquad ...(10)$$

with effective degrees of freedom = $\nu_p$

4.12   If an identical test has not previously been carried out, it is sometimes possible to obtain acceptable estimates of uncertainty due to random effect by interpolation or extrapolation from well-known statistical methods such as linear regression or other appropriate investigations.

## 5.0 TYPE B EVALUATION OF STANDARD UNCERTAINTY

5.1 Type B evaluation of standard uncertainty is obtained by means other than the statistical analysis of a series of observations. It is usually based on scientific judgement using all relevant information available, which may include:

- previous measurement data
- experience with, or general knowledge of the behaviour and property of relevant materials and instruments
- manufacturer's specification
- data provided in calibration and other reports and
- uncertainties assigned to reference data taken from data book

5.2 When considering Type B uncertainty, we have to convert the quoted uncertainty to a standard uncertainty. Different cases of how this is done are given below.

5.3 Convert a quoted uncertainty that is a stated multiple of an estimate standard deviation to a standard uncertainty by dividing the quoted uncertainty by the multiplier.

Example
A calibration report states that the measurement uncertainty is 3 mV at 3 standard deviations. The standard uncertainty is then 3 mV divided by 3 which gives 1 mV.

5.4 Convert a quoted uncertainty to a standard uncertainty from the knowledge of the probability distribution of the uncertainty. Divide the quoted uncertainty by a factor which depends on the probability distribution.

### 5.4.1 Rectangular Probability Distribution
It is used when uncertainties are given by maximum bound within which all values are equally probable. The standard uncertainty is computed by dividing the half-interval 'a' by $\sqrt{3}$.



Rectangular Distribution
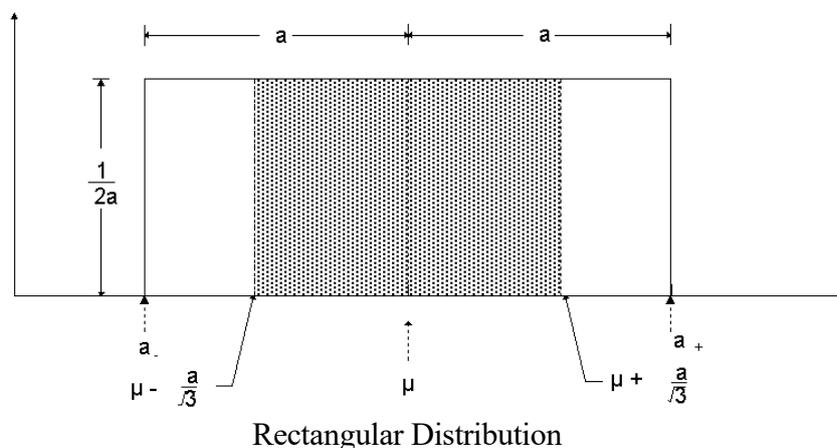
<u>Examples</u>

a. The measurement accuracy of a voltmeter is ± 0.05 %. The half-interval limit is 0.05 % and the standard uncertainty is given by:

$$u(V) = \frac{0.05\ \%}{\sqrt{3}} \qquad \qquad ...(11)$$

b. The resolution of a digital voltage display is 1 mV. Thus, the interval is 1 mV and the half-interval limit is half of 1 mV. The standard uncertainty is then given by:

$$u(R) = \frac{0.5}{\sqrt{3}}\ x\ 1\ mV \qquad \qquad ...(12)$$

c. The hysteresis effect of an instrument is 0.1 %. The difference between maximum and minimum for the same input gives the interval and the half-interval limit is half of 0.1 %. The standard uncertainty is given by:

$$u(H) = \frac{0.5}{\sqrt{3}}\ x\ 0.1\ \% \qquad \qquad ...(13)$$

d. The maximum drift of the value of a capacitance standard between calibration intervals is 0.001 pF. The history of the capacitance standard for the past few years showed that the capacitance value changed by not more than 0.001 pF. The standard uncertainty is then given by:

$$u(drift) = \frac{0.001\ pF}{\sqrt{3}} \qquad \qquad ...(14)$$

5.4.2 Triangular Probability Distribution

The triangular distribution is a better model if it is known that most of the values are likely to be near the centre of the distribution. The standard uncertainty is computed by dividing the half-interval 'a' by $\sqrt{6}$ .
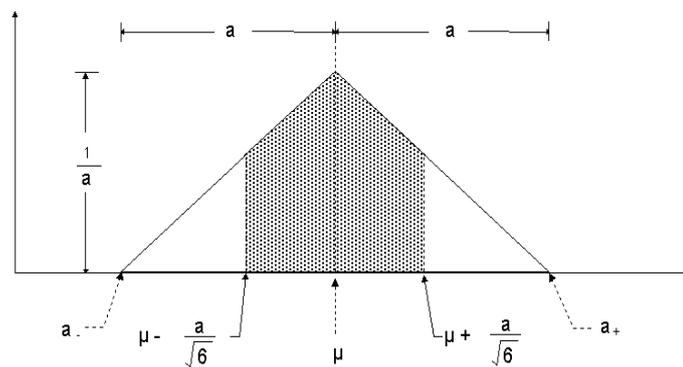


Triangular Distribution

The environmental temperature is controlled such that it is always near the centre of the range $20 \pm 2$ °C. The half-interval limit is then 2 °C and the standard uncertainty is given by:

$$u(T) = \frac{2 \ \text{°C}}{\sqrt{6}} \qquad \text{...(15)}$$

### 5.4.3 U-Shape Probability Distribution

This distribution is used typically in the case of the mismatch uncertainty in radio and microwave power frequency power measurement. At high frequency, the power is delivered from a source to a load, and reflection occurs when the impedances do not match. The mismatch uncertainty is given by $2\Gamma_S\Gamma_L$.

$\Gamma_S$ and $\Gamma_L$ are the reflection coefficients of the source and the load respectively. The standard uncertainty is given by:

$$u(m) = \frac{2 \ \Gamma_S \ \Gamma_L}{\sqrt{2}} \qquad \text{...(16)}$$
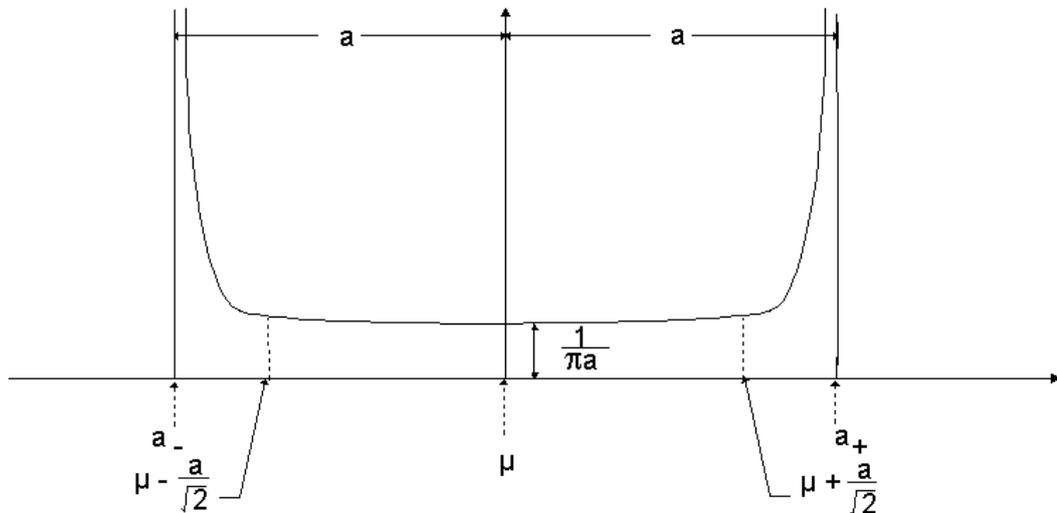


U-shape Distribution

Example

The output power of a signal generator is measured by a power meter. The magnitude of the reflection coefficients of the signal generator and the power meter is 0.2 and 0.091 respectively. The standard uncertainty due to the mismatch is given by:

$$u(m) = \frac{2 \ \text{x} \ 0.2 \ \text{x} \ 0.091}{\sqrt{2}} \qquad \text{...(17)}$$

5.4.4   Normal or Gaussian Probability Distribution
This distribution form can be assumed for an uncertainty that defines a confidence interval having a given level of confidence of say 95 % or 99 %.  The standard uncertainty is obtained by dividing the quoted uncertainty by the appropriate factor for such a distribution (refer to APPENDIX 2 on the choice of factors).
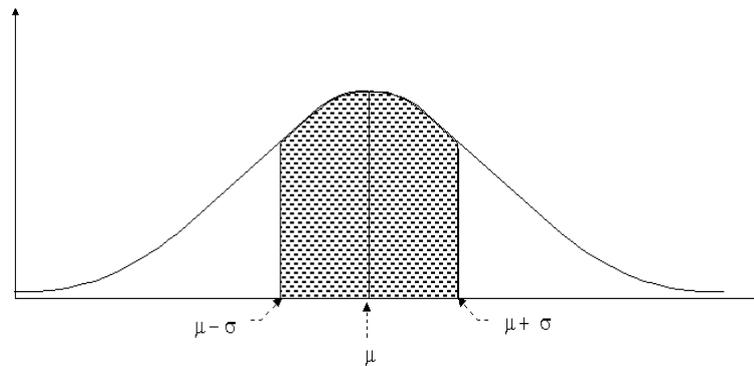


Gaussian Distribution

Examples

a.      A calibration report states that the uncertainty is ± 0.1 dB with a coverage factor of 2.63. The standard uncertainty is given by:

$$u(x) = \frac{0.1\ \text{dB}}{2.63} \qquad ...(18)$$

b.      The uncertainty specification on a DC 1 V range is given as ± 5 ppm at a confidence level of 99 %. Assuming normally distributed, the standard uncertainty is given by:

$$u(x_1) = \frac{5\ \mu\text{V}}{2.58} \qquad ...(19)$$

5.5     The rectangular distribution is a reasonable default model in the absence of any other information. But if it is known that the value of the quantity in question is near the centre of the limits, a triangular or a normal distribution may be a better model.

5.6     Type B uncertainty is obtained from a priori probability distributions. It is implicitly assumed that the probability distribution is exactly known. In most cases, we can assume that the degrees of freedom for such standard uncertainty as infinite. This is a reasonable assumption as it is a common practice to choose a Type B uncertainty such that the probability of the quantity in question lying outside the uncertainty band is extremely small. (Refer to APPENDIX 1 for treatment of degrees of freedom).

## 6.0    COMBINED STANDARD UNCERTAINTY

6.1    The estimated standard deviation associated with the output estimate or measurement result $y$, termed combined standard uncertainty and denoted by $u_c(y)$, is determined from the standard uncertainty $u(x_i)$ associated with each input estimate $x_i$ (refer to section 3 on 'Modelling the Measurement').

6.2    The following equations of computing the combined standard uncertainty are based on a first order Taylor series approximation of the equation (1). This method is often called the law of propagation of uncertainty.

6.3    When the non-linearity of $f$ is significant, higher-order terms in the Taylor series expansion must be included in the expression for $u_c^2(y)$.

6.4    Uncorrelated Input Quantities

6.4.1    In the case where all input quantities are statistically independent, their joint probability distributions are products of their individual probability distributions.

6.4.2    The combined standard uncertainty $u_c(y)$ is the positive square root of the combined variance $u_c^2(y)$ which is given by:

$$u_c^{\,2}(y) \;=\; \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^{\!2} u^2(x_i) \qquad\qquad ...(20)$$

where $f$ is the function given in equation (1)

6.4.3    The partial derivatives $\dfrac{\partial f}{\partial x_i}$ are called sensitivity coefficients. They describe how the output estimate $y$ varies with changes in the values of the input estimates $x_1, x_2, ... x_N$. The combined variance $u_c^2(y)$ can be viewed as a sum of terms. These terms represent the estimated variance associated with the output estimate $y$ generated by the estimated variance associated with each input estimate $x_i$.

6.4.4    Denoting $\dfrac{\partial f}{\partial x_i}$ by $c_i$, the equation (20) can be written as:

$$u_c^2(y) \;=\; \sum_{i=1}^{N} \left[ c_i \; u(x_i) \right]^2$$

$$=\; \sum_{i=1}^{N} u_i^2(y) \qquad\qquad ...(21)$$

where:  $u_i(y) = c_i \, u(x_i)$ and $c_i = \dfrac{\partial f}{\partial x_i}$

6.4.5 The sensitivity coefficients $\dfrac{\partial f}{\partial x_i}$ are sometimes determined experimentally. In this case, the knowledge of the function $f$ is accordingly reduced to an empirical first-order Taylor series expansion based on the measured sensitivity coefficients.

6.4.6 If $Y$ is of the form, ie. $Y = c\,X_1^{P_1}\ X_2^{P_2}\ ....\ X_N^{P_N}$ with the exponents $P_i$ known to be positive or negative numbers, the combined variance equation (20) can be expressed as:

$$\left[\frac{u_C(y)}{y}\right]^2 = \sum_{i=1}^{N}\left[\frac{p_i u(x_i)}{(x_i)}\right]^2 \qquad\qquad ...(22)$$

6.4.7 If each $P_i$ is either +1 or - 1, equation (22) becomes equation (23) which shows that for this special case the relative combined variance associated with the estimate $y$ is simply equal to the sum of the estimated relative variances associated with the input estimates $x_i$ .

$$\left[\frac{u_c(y)}{y}\right]^2 = \sum_{i=1}^{N}\left[\frac{u(x_i)}{(x_i)}\right]^2 \qquad\qquad ...(23)$$

6.5 Correlated Input Quantities

6.5.1 In the case where two or more input quantities are interdependent. The appropriate expression for the combined variance $u_c^2(y)$ associated with the result of a measurement is:

$$u_c^2(y) = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,\ x_j) \qquad\qquad ...(24)$$

$$u_c^2(y) = \sum_{i=1}^{N}\left[\frac{\partial f}{\partial x_i}\right]^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j) \qquad\qquad ...(25)$$

where $u(x_i,x_j)$ is the estimated covariance between $x_i$ and $x_j$

6.5.2 The interdependence of two variables is characterized by their coefficients.

$$r(x_i,x_j) = \frac{u(x_i,x_j)}{u(x_i)\,u(x_j)} \qquad\qquad ...(26)$$

6.5.3 With the aid of equation (26) and the sensitivity coefficients, $c_i$ described in the paragraph 6.4.3 and 6.4.4, the equation (25) becomes:

$$u_c^2(y) = \sum_{i=1}^{N}c_i^2\,u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}c_i\,c_j\,u(x_i)\,u(x_j)\,r(x_i,x_j) \qquad\qquad ...(27)$$

6.5.4    Correlation can occur if the same measurement is used more than once in the same measurement process, however, its effect on the combined uncertainty may be positive, i.e. the uncertainty is increased or negative which will lead to a reduction in the uncertainty.

6.5.5    If positive correlation is suspected but the correlation coefficient cannot be easily calculated then it is reasonable to assume a correlation coefficient of +1 and simply add the correlated uncertainty contributions.

6.5.6    If all of the input estimates are correlated with correlation coefficients $r(x_i, x_j) = +1$, equation (27) reduces to :

$$u_c^2(y) = \left[ \sum_{i=1}^{N} c_i\, u(x_i) \right]^2$$

$$= \left[ \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\, u(x_i) \right]^2 \qquad \text{...(28)}$$

6.5.7    "In the case of two randomly varying quantities $q$ and $r$, with their respective means $\bar{q}$ and $\bar{r}$ calculated from $n$ independent pairs of simultaneous observations of $q$ and $r$ made under the same conditions of measurement, the covariance is estimated by:"

$$s(\bar{q},\, \bar{r}) = \frac{1}{n(n-1)} \sum_{k=1}^{n} (q_k - \bar{q})\,(r_k - \bar{r}) \qquad \text{...(29)}$$

where $q$, and $r$ are the individual observations made simultaneously.

6.5.8    For two correlated input quantities $X_i$ and $X_j$, the correlation coefficient is obtained from equation (26).

$$r(x_i, x_j) = r(\overline{X}_i, \overline{X}_j)$$

$$= \frac{s(\overline{X}_i, \overline{X}_j)}{s(\overline{X}_i)\, s(\overline{X}_j)} \qquad \text{...(30)}$$

## 7.0  EXPANDED UNCERTAINTY

7.1  While the combined standard uncertainty $u_c(y)$ can be used to express the uncertainty of measurement results, many applications required a measure of uncertainty that defines an interval about the measurement result $y$ within which the value of the measurand $Y$ can be confidently asserted to lie.

7.2  The measure of uncertainty intended to meet this requirement is termed expanded uncertainty, denoted by symbol $U$ and is obtained by multiplying $u_c(y)$ by a coverage factor. Thus $U = k\,u_c(y)$ and it can be confidently asserted that $y - U \le Y \le y + U$, commonly written as $Y = y \pm U$.

Current international practice is to give a level of confidence of approximately 95 % (95.45 %). When level of confidence is fixed, $k$ varies with effective degrees of freedom. Although in many cases, $k$ equal to 2 can be used where effective degrees of freedom is reasonably large, greater or equal to 30. For cases where effective degrees of freedom is small, it is necessary to obtain the value of $k$ using the t-distribution table. The calculation of effective degrees of freedom is outlined in Appendix 1.

7.3  In certain cases, whereby $u_c(y)$ is dominated by one component of uncertainty derived from Type B evaluation, the use of $k$ value obtained from the t-distribution table is likely to give an interval corresponding to a level of confidence more than approximately 95 %. In this situation, knowledge of the assumed probability distribution of the measurand $Y$ is required to obtain the value of $k$ that produces an interval corresponding to a level of confidence of approximately 95 %.

## 8.0  REPORTING OF RESULTS

8.1  In stating the measurement result in the calibration report, the following information shall be provided.
   a.  the measurement result,
   b.  the expanded uncertainty,
   c.  the level of confidence used in defining the interval of the expanded uncertainty, and
   d.  the coverage factor, $k$ ( and if necessary, the effective degrees of freedom ) used in defining the interval of the expanded uncertainty.

Example
The measurement result is $Y = y \pm U$. The reported expanded uncertainty of measurement is stated as the standard uncertainty of measurement multiplied by the coverage factor $k = 2$, which is for a level of confidence of approximately 95 %.

8.2 The numerical value of the uncertainty of measurement should be given to at most two significant figures. If the rounding brings the numerical value of the uncertainty of measurement down by more than 5 %, the rounding up value should be used.

8.3 The numerical value of the measurement result $y$ should in the final statement normally be rounded to the least significant figure in the value of the expanded uncertainty assigned to the measurement result.

8.4 To round a number to $n$ significant figures, discard all digits to the right of the *nth* place. If the discarded number is less than one-half a unit in the *nth* place, leave the *nth* digit unchanged. If the discarded number is greater than one-half a unit in the nth place, increase the *nth* digit by 1. If the discarded number is exactly one-half a unit in the *nth* place, leave the *nth* digit unchanged if it is an even number and add 1 to it if it is odd.

Examples of rounding a number to four significant figures.
20.453 mm   20.45 mm
20.456 mm   20.46 mm
20.455 mm   20.46 mm
20.465 mm   20.46 mm


## 9.0 CONCLUSION

9.1 When evaluating the measurement uncertainty, it is important not to 'double count' uncertainty components. If a component of uncertainty arising from a particular effect is obtained from a Type B evaluation, it should be included as an independent component of uncertainty in the calculation of the combined standard uncertainty of the measurement result only to the extent that the effect does not contribute to the observed variability of the observations. This is because the uncertainty due to that portion of the effect that contributes to the observed variability is already included in the component of uncertainty obtained from statistical analysis of the observations.

9.2 Although this guide provides a framework for evaluating uncertainty, it cannot replace analytical thinking, skill and professionalism. The treatment of uncertainty is neither a routine task nor a purely mathematical one; it depends on detailed knowledge of the nature of the measurand and of the measurement system. The quality and usage of the measurement uncertainty quoted ultimately depends on the knowledge, critical investigation, and integrity of those who contribute to the assignment of the value.

## 10    REFERENCES

[1]    JCGM 100, Guide to the Expression of Uncertainty in Measurement

[2]    ISO/IEC 17025, General Requirements for the Competence of Testing and Calibration Laboratories

[3]    UKAS M3003, The Expression of Uncertainty and Confidence in Measurement

[4]    NIST Technical Note 1297, Guideline for Evaluating and Expressing the Uncertainty of NIST Measurement Results

[5]    JCGM 200, International Vocabulary of Metrology - Basic and General Concepts and associated terms (VIM)

[6]    EA-4/02, Evaluation of the Uncertainty of Measurement in calibration

## APPENDIX 1
## COVERAGE FACTOR

1     To estimate the value of such a coverage factor requires taking into account the uncertainty of $u_c(y)$, which is characterized by the "effective degrees of freedom," $v_{eff}$, of $u_c(y)$. For example, if $v_{eff}$ is less than 11, simply assuming that the uncertainty of $u_c(y)$ is negligible and taking $k = 2$ may be inadequate if an expanded uncertainty $U = k\, u_c(y)$ that defines an interval having a level of confidence close to 95 % is required for a specific application. In this case, and in other similar cases where $v_{eff}$ of $u_c(y)$ is comparatively small and an interval having a level of confidence close to a specified level is required, it is unlikely that the uncertainty of $u_c(y)$ would be considered negligible. Instead, the small value of $v_{eff}$, and thus the uncertainty of $u_c(y)$, would probably be taken into account when determining $k$.

2     The four-step procedure for calculating $k$ is as follows:

1)    Obtain $y$ and $u_c(y)$.

2)    Estimate the effective degrees of freedom $v_{eff}$ of $u_c(y)$ from the Welch-Satterthwaite formula:

$$v_{eff} = \frac{u_c^4(y)}{\displaystyle\sum_{i=1}^{N} \frac{c_i^4\, u^4(x_i)}{v_i}}$$

where $c_i = \dfrac{\partial f}{\partial x_i}$, all of the $u(x_i)$ are mutually statistically independent, $v_i$ is the degrees of freedom of $u(x_i)$ and

$$v_{eff} \le \sum_{i=1}^{N} v_i$$

The degrees of freedom of a standard uncertainty $u(x_i)$ obtained from Type A evaluation, where $x_i = X_i$ and $u(x_i) = s(X_i)$, is $v_i = n - 1$. If $m$ parameters are estimated by fitting a curve to an $n$ data points by the method of least squares, the degrees of freedom of the standard uncertainty of each parameter is $n$ - $m$.

The degrees of freedom associated with a standard uncertainty $u(x_i)$, obtained from a Type B evaluation with lower and upper limits $a_-$ and $a_+$, are set in such a way that the probability of the quantity in question lying outside these limits is in fact extremely small, the degrees of freedom may be taken to be $v_i \rightarrow \infty$. Otherwise, the following equation may be used to define the degrees of freedom, $v$, that is to be associated with a standard uncertainty $u(x_i)$ obtained from a Type B evaluation :

$$v \approx \frac{1}{2} \left[ \frac{u^2(x_i)}{\sigma^2 \left[ u(x_i) \right]} \right]$$

$$\approx \frac{1}{2} \left[ \frac{\Delta u(x_i)}{u(x_i)} \right]^{-2}$$

The quantity in large brackets is the relative uncertainty of $u(x_i)$, which for a Type B evaluation is a subjective quantity. This value is to be obtained from experience and knowledge of the measurement procedures.

Example :
Assume it is judged that the evaluation of $u(x_i)$ is reliable to about 25 %. Then it may be taken to mean that the relative uncertainty $\frac{\Delta u(x_i)}{u(x_i)} = 0.25$ and from the above equation

$$v = \frac{0.25^{-2}}{2} = 8$$

3) Obtain the $t$-factor $t_p(v_{eff})$ for the required level of confidence $p$ from a table of values of $t_p(v)$ from the Student's $t$-distribution, (given in APPENDIX 2.) If $v_{eff}$ is not an integer, which will usually be the case, either interpolate or truncate $v_{eff}$ to the next lower integer. Truncation of $v_{eff}$ to the next higher integer could be also possible provide there is enough evidence proof the uncertainty claimed is not underestimated.

4) Take $k = t_p(v_{eff})$ and calculate $U = k \, u_c(y)$.

## t-DISTRIBUTION

Value of $t_p(\nu)$ from the *t*-distribution for degrees of freedom $\nu$ that defines an interval $-t_p(\nu)$ to $+t_p(\nu)$ that encompasses the fraction *p* of the distribution

| Degrees of freedom $\nu$ | Fraction *p* in percent | | | | | |
|---|---|---|---|---|---|---|
| | 68.27[a] | 90.00 | 95.00 | 95.45[a] | 99.00 | 99.73[a] |
| 1 | 1.84 | 6.31 | 12.71 | 13.97 | 63.66 | 235.8 |
| 2 | 1.32 | 2.92 | 4.30 | 4.53 | 9.92 | 19.21 |
| 3 | 1.20 | 2.35 | 3.18 | 3.31 | 5.84 | 9.22 |
| 4 | 1.14 | 2.13 | 2.78 | 2.87 | 4.60 | 6.62 |
| 5 | 1.11 | 2.02 | 2.57 | 2.65 | 4.03 | 5.51 |
| 6 | 1.09 | 1.94 | 2.45 | 2.52 | 3.71 | 4.90 |
| 7 | 1.08 | 1.89 | 2.36 | 2.43 | 3.50 | 4.53 |
| 8 | 1.07 | 1.86 | 2.31 | 2.37 | 3.36 | 4.28 |
| 9 | 1.06 | 1.83 | 2.26 | 2.32 | 3.25 | 4.09 |
| 10 | 1.05 | 1.81 | 2.23 | 2.28 | 3.17 | 3.96 |
| | | | | | | |
| 11 | 1.05 | 1.80 | 2.20 | 2.25 | 3.11 | 3.85 |
| 12 | 1.04 | 1.78 | 2.18 | 2.23 | 3.05 | 3.76 |
| 13 | 1.04 | 1.77 | 2.16 | 2.21 | 3.01 | 3.69 |
| 14 | 1.04 | 1.76 | 2.14 | 2.20 | 2.98 | 3.64 |
| 15 | 1.03 | 1.75 | 2.13 | 2.18 | 2.95 | 3.59 |
| | | | | | | |
| 16 | 1.03 | 1.75 | 2.12 | 2.17 | 2.92 | 3.54 |
| 17 | 1.03 | 1.74 | 2.11 | 2.16 | 2.90 | 3.51 |
| 18 | 1.03 | 1.73 | 2.10 | 2.15 | 2.88 | 3.48 |
| 19 | 1.03 | 1.73 | 2.09 | 2.14 | 2.86 | 3.45 |
| 20 | 1.03 | 1.72 | 2.09 | 2.13 | 2.85 | 3.42 |
| | | | | | | |
| 25 | 1.02 | 1.71 | 2.06 | 2.11 | 2.79 | 3.33 |
| 30 | 1.02 | 1.70 | 2.04 | 2.09 | 2.75 | 3.27 |
| 35 | 1.01 | 1.70 | 2.03 | 2.07 | 2.72 | 3.23 |
| 40 | 1.01 | 1.68 | 2.02 | 2.06 | 2.70 | 3.20 |
| 45 | 1.01 | 1.68 | 2.01 | 2.06 | 2.69 | 3.18 |
| | | | | | | |
| 50 | 1.01 | 1.68 | 2.01 | 2.05 | 2.68 | 3.16 |
| 100 | 1.005 | 1.660 | 1.984 | 2.025 | 2.626 | 3.077 |
| $\infty$ | 1.000 | 1.645 | 1.960 | 2.000 | 2.576 | 3.000 |

[a] For a quantity *z* described by a normal distribution with expectation $\mu_z$ and standard deviation $\sigma$, the interval $\mu_z \pm k\sigma$ encompasses *p* = 68.27, 95.45, and 99.73 percent of the distribution for *k* = 1, 2 and 3, respectively.

# APPENDIX 3
## FLOWCHART FOR EVALUATING UNCERTAINTY OF MEASUREMENT

**FLOWCHART OF SUMMARY FOR EVALUATING UNCERTAINTY OF MEASUREMENT**

```
                          ┌─────────────┐
                          │    START    │
                          └─────────────┘
                                 │
                                 ▼
              ┌──────────────────────────────────────┐
              │ Mathematical Model                    │
              │ Derive or estimate the functional     │
              │ relationship between the measured     │
              │ quantity Y and its N input quantities │
              │ Xi.                                   │
              └──────────────────────────────────────┘
                                 │
                                 ▼
              ┌──────────────────────────────────────┐
              │ Obtain Input Quantities               │
              │ Determine xi, the estimated value of  │
              │ input quantity Xi, either by a series │
              │ of n observations or by other means.  │
              └──────────────────────────────────────┘
                                 │
                                 ▼
              ┌──────────────────────────────────────┐
              │ Uncertainty Equation                  │
              │ List all possible sources of error in │
              │ the measurement process and determine │
              │ the uncertainty equation.             │
              └──────────────────────────────────────┘
```

- **Mathematical Model** — Derive or estimate the functional relationship between the measured quantity $Y$ and its $N$ input quantities $X_i$.
- **Obtain Input Quantities** — Determine $x_i$, the estimated value of input quantity $X_i$, either by a series of $n$ observations or by other means.
- **Uncertainty Equation** — List all possible sources of error in the measurement process and determine the uncertainty equation.

Standard Uncertainty Evaluation

Type A → Type A Evaluation

Type B → Type B Evaluation

Standard Uncertainty Evaluation Completed ? — No / Yes

Calculate Combined Standard Uncertainty

Calculate Expanded Uncertainty

Reporting of Uncertainty

END

# APPENDIX 3
# FLOWCHART FOR EVALUATING UNCERTAINTY OF MEASUREMENT

FLOWCHART FOR EVALUATING TYPE A STANDARD UNCERTAINTY

Type A standard uncertainty evaluation

Decide statistical methods

Calculate the mean of $n$ readings

$$\bar{q} = \frac{1}{n}\sum_{k=1}^{n} q_k$$

Calculate the standard deviation of $n$ readings

$$s(q_k) = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}\left(q_k - \bar{q}\right)^2}$$

Calculate the standard deviation of the mean

$$s(\bar{q}) = \frac{s(q_k)}{\sqrt{n}}$$

Assign the standard deviation of the mean to Type A standard uncertainty

$$u(x_i) = s(\bar{q})$$

Calculate the degrees of freedom

$$v_i = n - 1$$

By pooled experimental standard deviation, linear regression, or other recognised statistical methods

Pooled experimental standard deviation method

Pooled experimental standard deviation

$$s_p = \sqrt{\frac{\sum_{i=1}^{M} v_i s_i^2}{\sum_{i=1}^{M} v_i}}$$

Degrees of freedom

$$v_p = \sum_{i=1}^{M} v_i$$

Type A standard uncertainty

$$u(x_i) = \frac{s_p}{\sqrt{n}}$$

Type A standard uncertainty evaluation
END

# APPENDIX 3
# FLOWCHART FOR EVALUATING UNCERTAINTY OF MEASUREMENT

## FLOWCHART FOR EVALUATING TYPE B STANDARD UNCERTAINTY

```
                    ┌──────────────────────┐
                    │   Type B standard    │
                    │ uncertainty evaluation│
                    │        START         │
                    └──────────────────────┘
                              │
                              ▼
          ┌──────────────────────────────────────┐
          │ Assign or estimate the uncertainty U or│
          │ half range of limits a, and probability│
          │ distribution of type B input quantity. │
          └──────────────────────────────────────┘
                              │
                              ▼
```

Calculate type B standard uncertainty based on probability distribution

$$\text{Normal}: \quad u(x_i) = \frac{U}{k}$$

$$\text{Rectangular}: \quad u(x_i) = \frac{a}{\sqrt{3}}$$

$$\text{U}-\text{shape}: \quad u(x_i) = \frac{a}{\sqrt{2}}$$

$$\text{Triangular}: \quad u(x_i) = \frac{a}{\sqrt{6}}$$

or others.

**Degrees of freedom provided?**

Yes → Obtain degrees of freedom $\nu_i$

No → **Probability of input quantities lying outside limits extremely low?**

Yes → Estimate degrees of freedom $\nu_i = \infty$

No → Estimate degrees of freedom
$$\nu_i = \frac{1}{2}\left[\frac{\Delta u(x_i)}{u(x_i)}\right]^{-2}$$

```
                    ┌──────────────────────┐
                    │   Type B standard    │
                    │ uncertainty evaluation│
                    │         END          │
                    └──────────────────────┘
```

## APPENDIX 3
## FLOWCHART FOR EVALUATING UNCERTAINTY OF MEASUREMENT

## FLOWCHART FOR COMBINING UNCERTAINTY

```
                    ┌─────────────────┐
                    │ Calculate combined
                    │   uncertainty
                    │     START
                    └─────────────────┘
                             │
                             ▼
                    ◇ Input quantities correlated? ◇
         No ────────┘                      └──────── Yes
```

**(Left branch — No)**

Calculate the sensitivity coefficients for each of the input quantities

$$c_i = \frac{\partial f}{\partial x_i}$$

Calculate the combined uncertainty for uncorrelated input quantities

$$u_c(y) = \sqrt{\sum_{i=1}^{N}\{c_i u(x_i)\}^2} = \sqrt{\sum_{i=1}^{N}\{u_i(y)\}^2}$$

**(Right branch — Yes)**

Calculate the sensitivity coefficients for each of the input quantities

$$c_i = \frac{\partial f}{\partial x_i}$$

Calculate or estimate the correlation coefficients

$$r(x_i, x_j) = \frac{s(\bar{X}_i, \bar{X}_j)}{s(\bar{X}_i)s(\bar{X}_j)}$$

Calculate the combined standard uncertainty for correlated input quantities

$$u_c(y) = \sqrt{\sum_{i=1}^{N}[c_i u(x_i)]^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} c_i c_j u(x_i)\, u(x_j)\, r(x_i, x_j)}$$

```
                    ┌─────────────────┐
                    │ Calculate combined
                    │   uncertainty
                    │      END
                    └─────────────────┘
```

**APPENDIX 3**
**FLOWCHART FOR EVALUATING UNCERTAINTY OF MEASUREMENT**


**FLOWCHART FOR CALCULATING EXPANDED UNCERTAINTY**

```
         ╭──────────────────────╮
         │  Calculate expanded  │
         │     uncertainty      │
         │        START         │
         ╰──────────────────────╯
                    │
                    ▼
  ┌─────────────────────────────────────────┐
  │ Calculate the effective degrees of freedom │
  │ of the combined standard uncertainty      │
  │                                           │
```

$$\nu_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{N} \frac{[c_i u(x_i)]^4}{\nu_i}}$$

$$= \frac{u_c^4(y)}{\sum_{i=1}^{N} \frac{u_i^4(y)}{\nu_i}}$$

```
  └─────────────────────────────────────────┘
                    │
                    ▼
  ┌─────────────────────────────────────────┐
  │ Obtain the coverage factor k from the "t" │
  │ distribution table at the appropriate degrees │
  │ of freedom and level of confidence.       │
  └─────────────────────────────────────────┘
                    │
                    ▼
  ┌─────────────────────────────────────────┐
  │ Calculate the expanded uncertainty        │
  │                                           │
```

$$U = k\, u_c(y)$$

```
  └─────────────────────────────────────────┘
                    │
                    ▼
         ╭──────────────────────╮
         │  Calculate expanded  │
         │ uncertainty evaluation │
         │         END          │
         ╰──────────────────────╯
```

| | |
|---|---|
| $a$ | Half-width of a rectangular distribution of possible values of input quantity $X_i$ : $a = (a_+ - a_-)/2$ |
| $a_+$ | Upper bound, or upper limit, of input quantity $X_i$ |
| $a_-$ | Lower bound, or lower limit, of input quantity $X_i$ |
| $c_i$ | Partial derivative or sensitivity coefficient: $c_i \equiv \partial f / \partial x_i$ |
| $f$ | Functional relationship between measurand $Y$ and input quantities $X_i$ on which $Y$ depends, and between output estimate $y$ and input estimates $x_i$ on which $y$ depends |
| $\partial f / \partial x_i$ | Partial derivative with respect to input quantity $X_i$ of functional relationship $f$ between measurand $Y$ and input quantities $X_i$ on which $Y$ depends, evaluated with estimates $x_i$ for the $X_i$ : $$\partial f / \partial x_i = \partial f / \partial X_i \big|_{x_1, x_2, \cdots, x_N}$$ |
| $k$ | Coverage factor used to calculate expanded uncertainty $U = k\, u_c(y)$ of output estimate $y$ from its combined standard uncertainty $u_c(y)$, where $U$ defines an interval $Y = y \pm U$ having a high level of confidence |
| $n$ | Number of repeated observations |
| $N$ | Number of input quantities $X_i$ on which measurand $Y$ depends |
| $P$ | Probability; level of confidence: $0 \leq p \leq 1$ |
| $q$ | Randomly varying quantity described by a probability distribution |
| $\bar{q}$ | Arithmetic mean or average of $n$ independent repeated observations $q_k$ of randomly-varying quantity $q$; estimate of the expectation or mean $\mu_q$ of the probability distribution of $q$ |
| $q_k$ | $k$th independent repeated observation of randomly-varying quantity $q$ |
| $r(x_i, x_j)$ | Estimated correlation coefficient associated with input estimates $x_i$ and $x_j$ that estimate input quantities $X_i$ and $X_j$ : $r(x_i, x_j) = u(x_i, x_j)/u(x_i)u(x_j)$ |

# APPENDIX 4
# GLOSSARY OF TERMS

| | |
|---|---|
| $r(\overline{X}_i, \overline{X}_j)$ | Estimated correlation coefficient of input means $\overline{X}_i$ and $\overline{X}_j$, determined from $n$ independent pairs of repeated simultaneous observations $X_{i,k}$ and $X_{j,k}$ of $X_i$ and $X_j$; $r(\overline{X}_i, \overline{X}_j) = s(\overline{X}_i, \overline{X}_j) / s(\overline{X}_i) s(\overline{X}_j)$ |
| $s_P^2$ | Combined or pooled estimate of variance |
| $s_P$ | Pooled experimental standard deviation, equal to the positive square root of $s_P^2$ |
| $s^2(\overline{q})$ | Experimental variance of the mean $\overline{q}$; estimate of the variance $\sigma^2/n$ of $\overline{q} : s^2(\overline{q}) = s^2(q_k)/n$ ; estimated variance obtained from a Type A evaluation |
| $s(\overline{q})$ | Experimental standard deviation of the mean $\overline{q}$, equal to the positive square root of $s^2(\overline{q})$; $s(\overline{q})$ is a biased estimator of $\sigma(\overline{q})$; standard uncertainty obtained from a Type A evaluation |
| $s^2(q_k)$ | Experimental variance determined from $n$ independent repeated observations $q_k$ of $q$; estimate of the variance $\sigma^2$ of the probability distribution of $q$ |
| $s(q_k)$ | Experimental standard deviation, equal to the positive square root of $s^2(q_k)$; $s(q_k)$ is a biased estimator of the standard deviation $\sigma$ of the probability distribution of $q$ |
| $s(\overline{q}, \overline{r})$ | Estimate of the covariance of means $\overline{q}$ and $\overline{r}$ that estimate the expectations $\mu_q$ and $\mu_r$ of two randomly-varying quantities $q$ and $r$, determined from $n$ independent pairs of repeated simultaneous observations $q_k$ and $r_k$ of $q$ and $r$; estimated covariance obtained from a Type A evaluation |
| $s(\overline{X}_i, \overline{X}_j)$ | Estimate of the covariance of input means $\overline{X}_i$ and $\overline{X}_j$, determined from $n$ independent pairs of repeated simultaneous observations $X_{i,k}$ and $X_{j,k}$ of $X_i$ and $X_j$; estimated covariance obtained from a Type A evaluation |
| $t_p(\nu)$ | t-factor from the t-distribution for $\nu$ degrees of freedom corresponding to a given probability $p$ |
| $t_p(\nu_{eff})$ | t-factor from the t-distribution for $\nu_{eff}$ degrees of freedom corresponding to a given probability $p$, used to calculate expanded uncertainty $U_p$ |

$u^2(x_i)$ — Estimated variance associated with input estimate $x_i$ that estimates input quantity $X_i$

NOTE : When $x_i$ is determined from the arithmetic mean or average of $n$ independent repeated observations, $u^2(x_i) = s^2(\overline{X}_i)$ is an estimated variance obtained from a Type A evaluation

$u(x_i)$ — Standard uncertainty of input estimate $x_i$ that estimates input quantity $X_i$, equal to the positive square root of $u^2(x_i)$

NOTE: When $x_i$ is determined from the arithmetic mean or average of $n$ independent repeated observations, $u(x_i) = s(\overline{X}_i)$ is a standard uncertainty obtained from a Type A evaluation

$u(x_i, x_j)$ — Estimated covariance associated with two input estimates $x_i$ and $x_j$ that estimate input quantities $X_i$ and $X_j$

NOTE: When $x_i$ and $x_j$ are determined from $n$ independent pairs of repeated simultaneous observations, $u(x_i, x_j) = s(\overline{X}_i, \overline{X}_j)$ is an estimated covariance obtained from a Type A evaluation

$u_c^2(y)$ — Combined variance associated with output estimate $y$

$u_c(y)$ — Combined standard uncertainty of output estimate $y$, equal to the positive square root of $u_c^2(y)$

$u(x_i)/|x_i|$ — Relative standard uncertainty of input estimate $x_i$

$[u(x_i)/x_i]^2$ — Estimated relative variance associated with input estimate $x_i$

$U$ — Expanded uncertainty of output estimate $y$ that defines an interval $Y = y \pm U$ having a high level of confidence, equal to coverage factor $k$ times the combined standard uncertainty $u_c(y)$ of $y$ : $U = k\,u_c(y)$

$x_i$ — Estimate of input quantity $X_i$

NOTE: When $x_i$ is determined from the arithmetic mean or average of $n$ independent repeated observations, $x_i = \overline{X}_i$

$X_i$ — $i$th input quantity on which measurand $Y$ depends

NOTE: $X_i$ may be the physical quantity or the random variable

$\overline{X}_i$ — Estimate of the value of input quantity $X_i$, equal to the arithmetic mean or average of $n$ independent repeated observations $X_{i,k}$ of $X_i$

$X_{i,k}$ — $k$th independent repeated observation of $X_i$

y — Estimate of measurand Y; result of a measurement; output estimate

Y — A measurand

$\dfrac{\Delta u(x_i)}{u(x_i)}$ — Estimated relative uncertainty of standard uncertainty $u(x_i)$ of input estimate $x_i$

$\mu_q$ — Expectation or mean of the probability distribution of randomly-varying quantity $q$

$\nu$ — Degrees of freedom (general)

$\nu_i$ — Degrees of freedom, or effective degrees of freedom, of standard uncertainty $u(x_i)$ of input estimate $x_i$

$\nu_{eff}$ — Effective degrees of freedom of $u_c(y)$, used to obtain $t_p(\nu_{eff})$ for calculating expanded uncertainty $U_p$

$\sigma^2$ — Variance of a probability distribution of (for example) a randomly-varying quantity $q$, estimated by $s^2(q_k)$

$\sigma$ — Standard deviation of a probability distribution, equal to the positive square root of $\sigma^2$; $s(q_k)$ is a biased estimator of $\sigma$

$\sigma^2(\overline{q})$ — Variance of $\overline{q}$, equal to $\sigma^2/n$, estimated by $s^2(\overline{q}) = s^2(q_k)/n$

$\sigma(\overline{q})$ — Standard deviation of $\overline{q}$, equal to the positive square root of $\sigma^2(\overline{q})$; $s(\overline{q})$ is a biased estimator of $\sigma(\overline{q})$

$\sigma^2\big[s(\overline{q})\big]$ — Variance of experimental standard deviation $s(\overline{q})$ of $\overline{q}$

$\sigma\big[s(\overline{q})\big]$ — Standard deviation of experimental standard deviation $s(\overline{q})$ of $\overline{q}$, equal to the positive square root of $\sigma^2\big[s(\overline{q})\big]$

## 12 WORKED EXAMPLES

There is a total of fourteen examples in this guide.

EG 1: Digital thermometer calibration
EG 2: Temperature measurement using thermocouple
EG 3: Calibration of platinum resistance thermometer (PRT)
EG 4: DC current measurement
EG 5: Calibration of the DC 20V range of a multimeter
EG 6: Harmonic measurement
EG 7: Power reference
EG 8: Transition time
EG 9: Calibration of an industrial pressure gauge
EG 10: Torque tester calibration
EG 11: Calibration of weighing machine
EG 12: Coordinate measuring machine measurement
EG 13: Micrometer calibration
EG 14: Illumination meter calibration

These examples are used to demonstrate and teach and are not to be considered as definitive analyses for these parameters. The uncertainty contributions in the examples are not defined or recommended or should be since these are dependent on the equipment used and the method of test.

# EXAMPLE 1 : DIGITAL THERMOMETER CALIBRATION

## 1.0     INTRODUCTION

Digital thermometers are widely used in industry to measure the electromotive force (emfs) produced by thermocouples. Modern digital thermometer comes with automatic cold junction compensation (ACJC) with temperature display. A typical schematic circuit diagram used in the calibration of digital thermometer is shown in Figure 1. A reference temperature is set up using the emf source and thermocouple wire. The reference temperature is compared against the temperature indicated in the display of the digital thermometer so as to obtain the correction on the digital thermometer. The calculation of the expanded uncertainty associated with the correction for digital thermometer measuring Type K thermocouple is outlined in this example.
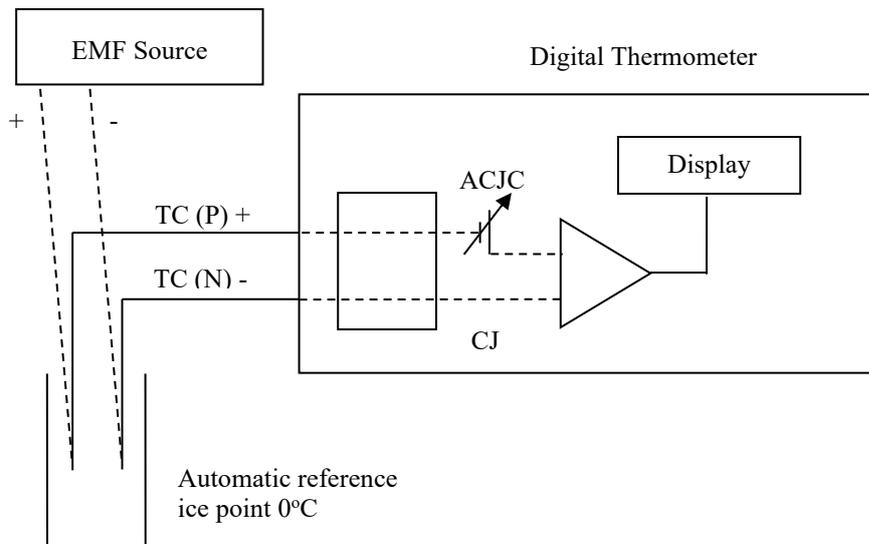
Figure 1: Schematic Diagram for Calibration of Digital Thermometer

where
TC(P)*          : Positive leg of the thermocouple
TC(N)*          : Negative leg of the thermocouple
CJ                  : Cold junction of the digital thermometer maintained at ambient temperature

Note :  *Digital thermometer can often be configured to measure different types of thermocouples.  The type of thermocouple used in the calibration schematic diagram must be the same as the type configured for measurement in the digital thermometer.

### EXAMPLE 1 : DIGITAL THERMOMETER CALIBRATION

### 2.0    MATHEMATICAL MODEL

The correction of the digital thermometer is given by the reference temperature minus the displayed temperature of the digital thermometer.

$$C = t_R - t_D \qquad \qquad ...(1)$$

where,
$C$ : Correction of the digital thermometer.
$t_R$ : Reference temperature.
$t_D$ : Displayed temperature of the digital thermometer.

### 3.0    UNCERTAINTY EQUATION

The combined standard uncertainty for the correction is given by equation (3). The partial derivatives are equal to one.

$$u_c(C) = \left[ \left( \frac{\partial C}{\partial t_R} \right)^2 u^2(t_R) + \left( \frac{\partial C}{\partial t_D} \right)^2 u^2(t_D) \right]^{\frac{1}{2}} \qquad ...(2)$$

$$u_c(C) = \left[ u^2(t_R) + u^2(t_D) \right]^{\frac{1}{2}} \qquad \qquad ...(3)$$

where,
$u_c(C)$ : Combined standard uncertainty for correction.
$u(t_R)$ : Standard uncertainty for the reference temperature.
$u(t_D)$ : Standard uncertainty for the displayed temperature.

### 4.0    STANDARD UNCERTAINTY EVALUATION

4.1    Determination of the standard uncertainty of the reference temperature $u(t_R)$

The thermocouple wires used in the calibration circuit (Figure 1) have to be calibrated separately since it's emf output may deviate from the International Thermocouple Reference Table values. The calibration of the Type K thermocouple wires is reported as having a correction of 0.0097 mV with an expanded uncertainty of ± 0.0015 mV at a confidence level of approximately 95 %, with a coverage factor $k$ equals to 2.

**EXAMPLE 1 : DIGITAL THERMOMETER CALIBRATION**

The reference ice point (Figure 1) has to be calibrated separately. The ice point has a zero correction with an expanded uncertainty of $\pm 0.02\ °C$ at a confidence level of approximately 95 %, with a coverage factor $k$ equals to 2.

The reference emf $E_{REF}$ is given by equation (4).

$$E_{REF} = E_{SOURCE} - \Delta E_{WIRE} - C_0\ \Delta t_{ICE} \qquad \qquad ...(4)$$

where,

$E_{REF}$ : Reference emf equivalent to a particular ITS-90 temperature.

$E_{SOURCE}$ : Emf output by the emf source.

$\Delta E_{WIRE}$ : Emf correction for the calibrated thermocouple wires.

$\Delta t_{ICE}$ : Temperature correction for the calibrated reference ice-point.

$C_0$ : Sensitivity coefficient for Type K thermocouple at the reference temperature of 0 °C, 0.0395 mV/ °C.

The standard uncertainty of the reference emf ($E_{REF}$) is given by equation (5).

$$u(E_{REF}) = [u^2(E_{SOURCE}) + u^2(\Delta E_{WIRE}) + C_0^2\ u^2(\Delta t_{ICE})]^{1/2} \qquad ...(5)$$

where,

$u(E_{REF})$ : Standard uncertainty of the reference emf.

$u(E_{SOURCE})$ : Standard uncertainty of the emf source.

$u(\Delta E_{WIRE})$ : Standard uncertainty of the correction for the thermocouple wires.

$u(\Delta t_{ICE})$ : Standard uncertainty of the correction for the reference ice point.

The emf source used in the calibration has the following one year specifications from the manufacturer.

$$E_{SOURCE}\ \leq 22\ mV \quad : \pm 0.0020\ mV$$
$$22\ mV < E_{SOURCE}\ < 50\ mV \quad : \pm 0.0027\ mV$$

# EXAMPLE 1 : DIGITAL THERMOMETER CALIBRATION

The uncertainty budget table for the reference emf is given in Table 1.

**Table 1**

| Source of Uncertainty | Symbol $X_I$ | Type | Uncertainty value (mV/°C) | Probability distribution | Coverage factor | Standard uncertainty ($u_i$) (mV) | Sensitivity coefficient $c_i$ | $\left|c_i\right| \times u(x_i)$ (mV) | Degrees of freedom $v$ |
|---|---|---|---|---|---|---|---|---|---|
| ***Emf Source*** | | | | | | | | | |
| $E_{SOURCE} \leq 22$ mV | $E_{SOURCE}$ | B | 0.0020 | Rectangular | 1.732 | 0.0012 | 1 | 0.0012 | ∞ |
| $22$ mV $< E_{SOURCE} < 50$ mV | | B | 0.0027 | Rectangular | 1.732 | 0.0016 | 1 | 0.0016 | ∞ |
| TC wires calibration | $\Delta E_{WIRE}$ | B | 0.0015 | Normal | 2.000 | 0.00075 | 1 | 0.00075 | ∞ |
| Reference ice point | $\Delta t_{ICE}$ | B | 0.02 | Normal | 2.000 | 0.01 | 0.0395 | 0.000395 | ∞ |

The standard uncertainty $u(E_{REF})$ of the reference emf is evaluated using equation (5).

for $E_{SOURCE} \leq 22$ mV ,

$$u(E_{REF}) = \left(0.0012^2 + 0.00075^2 + 0.000395^2\right)^{1/2}$$
$$= 0.0015 \text{ mV}$$

for $22$ mV $< E_{SOURCE} < 50$ mV ,

$$u(E_{REF}) = \left(0.0016^2 + 0.00075^2 + 0.000395^2\right)^{1/2}$$
$$= 0.0019 \text{ mV}$$

The effective degrees of freedom for each of the standard uncertainty is infinity.

The International Thermocouple Reference Table gives the relationship between the temperature and emf. The standard uncertainty of reference temperature $u(t_R)$ can be expressed in terms of the standard uncertainty of the reference emf, $u(E_{REF})$ .

$$u(t_R) = \left(\frac{\partial T}{\partial E}\right)_{E_{REF}} u(E_{REF}) \qquad \qquad \text{...(6)}$$

**EXAMPLE 1 : DIGITAL THERMOMETER CALIBRATION**

The $\left(\dfrac{dT}{dE}\right)_{E_{REF}}$ values are calculated from Type K International Thermocouple Reference Table.

For emf equal or less than 22 mV (corresponding to reference temperature from the range –200°C to 500°C), the maximum calculated $\left(\dfrac{dT}{dE}\right)_{E_{REF}}$ is $\dfrac{1}{0.015}$ (°C/mV). Whereas, for reference emf greater than 22 mV (corresponding to reference temperature of 600 °C to 1200 °C), the maximum calculated $\left(\dfrac{dT}{dE}\right)_{E_{REF}}$ is $\dfrac{1}{0.036}$ (°C/mV).

The standard uncertainty of reference temperature $u(t_R)$ is given in Table 2. This is evaluated using Equation 6.

**Table 2**

| Source of Uncertainty | $u(E_{REF})$ | $c_i = \left(\dfrac{\partial T}{\partial E}\right)_{E_{REF}}$ | $u(t_R)$ | Degrees of freedom $\nu_i$ |
|---|---|---|---|---|
| | (mV) | (°C/mV) | (°C) | |
| Reference emf $E_{REF} \leq 22$ mV (-200 °C to 500 °C) | 0.0015 | $\dfrac{1}{0.015}$ | 0.100 | $\infty$ |
| Reference emf $22$ mV $< E_{REF} < 50$ mV (600 °C to 1200 °C) | 0.0019 | $\dfrac{1}{0.036}$ | 0.053 | $\infty$ |

4.2     Determination of the standard uncertainty in the displayed temperature $u(t_D)$

With a high resolution emf source, it is possible to change the emf signal over a small range of values without producing a change in the displayed reading on the digital thermometer. As a result, there is a range of emf signals to the digital thermometer that gives the same displayed reading. This is one source of uncertainty of a digital instrument due to the resolution of its indicating device.

The maximum value $(a_+)$ and minimum value $(a_-)$ of the emf signals that give the same indication on the digital thermometer are averaged to give the nominal emf value. This can be experimentally performed by slowly approaching each chosen scale mark in one direction and then repeat by slowly approaching from the other direction. The maximum size of this range $(a_+ - a_-)$ is usually equals to the resolution of the digital thermometer $(\delta a)$.

**EXAMPLE 1 : DIGITAL THERMOMETER CALIBRATION**

If this technique is used in the calibration, the probability that the nominal emf value lies within the interval a- to a+ for all practical purposes is equal to one and the probability that the nominal emf value lies outside this interval is essentially zero. It is assumed that it is equally probable for the nominal emf to lie anywhere within the interval and therefore, a rectangular distribution is assumed with a half width of $\dfrac{\delta a}{2}$. The standard uncertainty in the displayed reading is given by equation (7).

$$u(t_D) = \left( \frac{\delta a}{2} \right) \left( \frac{1}{\sqrt{3}} \right) \qquad \qquad ...(7)$$

For resolution of 0.1 °C and 1.0 °C, $u(t_D)$ is equal to 0.029 °C and 0.29 °C respectively. The degrees of freedom associated with the standard uncertainty of the displayed temperature $u(t_D)$ is infinity.

**5.0    CALCULATION OF THE COMBINED STANDARD UNCERTAINTY $u_c(C)$ AND EXPANDED UNCERTAINTY $U$**

The calculations of the combined standard uncertainty and expanded uncertainty associated with the correction for Type K thermocouple are outlined in Table 3. The degrees of freedom of both the standard uncertainties of the reference temperature $u(t_R)$ and the displayed temperature $u(t_D)$ are equal to infinity. As a result, the degrees of freedom for the combined standard uncertainty $u_c(C)$ is also equal to infinity. This gives a coverage factor $k$ equals to 2 at a confidence level of approximately 95 %.

**Table 3**

| Range | $u(t_R)$ (°C) | $u(t_D)$ (°C) | $u_c(C)$ (°C) | $U=k\,u_c(C)$ (°C) |
|---|---|---|---|---|
| -200 °C to 500 °C with 0.1 °C resolution 1 °C resolution | 0.100 0.100 | 0.029 0.29 | 0.105 0.307 | 0.21 0.62 |
| 600 °C to 1200 °C with 0.1 °C resolution 1 °C resolution | 0.053 0.053 | 0.029 0.29 | 0.061 0.295 | 0.13 0.59 |

The expanded uncertainty associated with the correction is given by $U = k\,u_c(C)$ in Table 3 at a confidence level of approximately 95 % with coverage factor $k$ equals to 2.

# EXAMPLE 2 : TEMPERATURE MEASUREMENT USING THERMOCOUPLE

## 1.0    INTRODUCTION

A digital thermometer with a Type K thermocouple are used to measure the temperature inside a temperature chamber.  The temperature controller of the chamber is set at 400 °C.



Temperature controller set at 400 °C
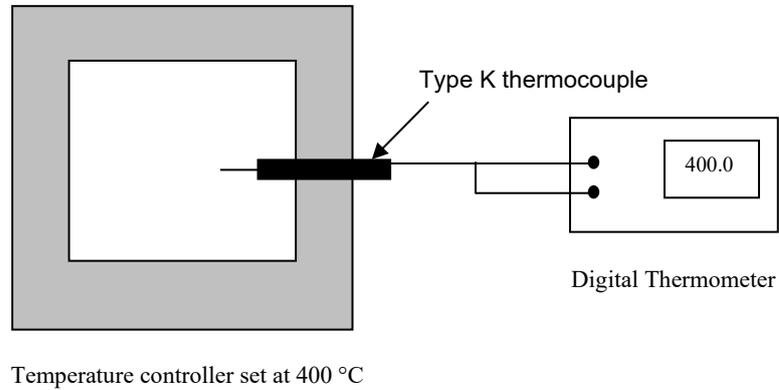
Figure 1 : Measurement set-up

1.1    Digital thermometer specification
Resolution : 0.1 °C
Uncertainty (one year) : $\pm 0.6$ °C

1.2    Thermocouple
The calibration report of the Type K thermocouple gives an uncertainty of $\pm 1.0$ °C at confidence level of approximately 95 % with a coverage factor $k$ equals to 2.0.  The correction for the thermocouple at 400 °C is 0.5 °C.

1.3    Measurement record
When the temperature chamber indicator reached 400 °C, the readings are taken after a stabilisation time of half an hour. Ten measurements are taken as given in Table 1.

**Table 1**

| Measurement ( i ) | $T_i$(°C) |
|---|---|
| 1 | 400.1 |
| 2 | 400.0 |
| 3 | 400.1 |
| 4 | 399.9 |
| 5 | 399.9 |
| 6 | 400.0 |
| 7 | 400.1 |
| 8 | 400.2 |
| 9 | 400.0 |
| 10 | 399.9 |

# EXAMPLE 2 : TEMPERATURE MEASUREMENT USING THERMOCOUPLE

## 2.0    MATHEMATICAL MODEL

The test temperature $t_x$ is given by:

$$t_x \cong t_{rdg} + \Delta t_{tc} + \Delta t_{imm} + \Delta t_{drift} + \Delta t_{ind} + \Delta t_{res} \qquad \text{...(1)}$$

where
$t_{rdg}$      : temperature reading of the Type K thermocouple.
$\Delta t_{tc}$     : temperature correction of the Type K thermocouple reading based on its calibration data.
$\Delta t_{imm}$   : temperature correction due to immersion error of the Type K thermocouple.
$\Delta t_{drift}$   : temperature correction due to drift of the Type K thermocouple.
$\Delta t_{ind}$    : temperature correction due to deviation of the digital thermometer.
$\Delta t_{res}$    : temperature correction due to the resolution of the digital thermometer.

## 3.0    UNCERTAINTY EQUATION

The combined standard uncertainty $u_c(t_x)$ of the test temperature is given by:

$$u_c(t_x) \cong \left[ \ u^2(t_{rdg}) + u^2(\Delta t_{tc}) + u^2(\Delta t_{imm}) + u^2(\Delta t_{drift}) + u^2(\Delta t_{ind}) + u^2(\Delta t_{res}) \ \right]^{1/2} \quad \text{...(2)}$$

where
$u(t_{rdg})$      : standard uncertainty of the Type K thermocouple reading.
$u(\Delta t_{tc})$      : standard uncertainty of the Type K thermocouple correction.
$u(\Delta t_{imm})$  : standard uncertainty of the Type K thermocouple immersion correction.
$u(\Delta t_{drift})$  : standard uncertainty of the Type K thermocouple drift correction.
$u(\Delta t_{ind})$   : standard uncertainty of the digital thermometer deviation correction.
$u(\Delta t_{res})$   : standard uncertainty of the digital thermometer resolution correction.

## 4.0    STANDARD UNCERTAINTY EVALUATION

### 4.1    Type A evaluation

### 4.1.1    Standard uncertainty of the thermocouple reading $u(t_{rdg})$

$$Mean\ value, \overline{T} = \frac{1}{10}\sum_{i=1}^{10} T_i = 400.02\ °C$$

Where $T_i$ are the ten measurements taken as listed in Table 1.

**EXAMPLE 2 : TEMPERATURE MEASUREMENT USING THERMOCOUPLE**

Standard deviation of the mean :

$$s\left(\overline{T}\right) = \frac{s(T_i)}{\sqrt{n}} = \frac{0.103°\,C}{\sqrt{10}} = 0.033°\,C$$

Thus, the standard uncertainty of the thermocouple reading $u(t_{rdg})$ is 0.033 °C.

Degrees of freedom : $\nu$ = n - 1 = 10 - 1 = 9

4.2     Type B evaluation

4.2.1   Standard uncertainty of the thermocouple correction $u(\Delta t_{tc})$

The correction for the thermocouple reading is 0.5 °C. The standard uncertainty of thermocouple correction $u(\Delta t_{tc})$ is :

$$u(\Delta t_{tc}) = \frac{1.0}{2.0} = 0.500°\,C$$

Degrees of freedom : $\nu = \infty$

4.2.2   Standard uncertainty of the thermocouple immersion correction $u(\Delta t_{imm})$

The uncertainty limit of the thermocouple immersion correction is ± 0.1 °C. Assuming a rectangular distribution, standard uncertainty of the thermocouple immersion correction $u(\Delta t_{imm})$ is,

$$u(\Delta t_{imm}) = \frac{0.1}{\sqrt{3}} = 0.058 \text{ °C}$$

Degrees of freedom : $\nu = \infty$

4.2.3   Standard uncertainty of the thermocouple drift correction $u(\Delta t_{drift})$

The uncertainty limit of the drift is ± 0.2°C. Assuming a rectangular distribution, standard uncertainty of the thermocouple drift correction $u(\Delta t_{drift})$ is,

$$u(\Delta t_{drift}) = \frac{0.2}{\sqrt{3}} = 0.115 \text{ °C}$$

Degrees of freedom : $\nu = \infty$

**EXAMPLE 2 : TEMPERATURE MEASUREMENT USING THERMOCOUPLE**

4.2.4    Standard uncertainty of the digital thermometer deviation correction $u(\Delta t_{ind})$

From specification, the uncertainty limit of the digital thermometer is ± 0.6 °C. Assuming a rectangular distribution, the standard uncertainty of the digital thermometer deviation correction $u(\Delta t_{ind})$ is,

$$u(\Delta t_{ind}) = \frac{0.6}{\sqrt{3}} = 0.346 \ ^{\circ}C$$

Degrees of freedom : $\nu = \infty$

4.2.5    Standard uncertainty of the digital thermometer resolution correction $u(\Delta t_{res})$

The half limit due to the resolution of the digital thermometer is 0.05 °C. Assuming a rectangular distribution, the standard uncertainty of the digital thermometer resolution correction $u(\Delta t_{res})$ is,

$$u(\Delta t_{ind}) = \frac{0.05}{\sqrt{3}} = 0.029 \ ^{\circ}C$$

Degrees of freedom : $\nu = \infty$

## 5.0    UNCERTAINTY BUDGET TABLE

| Source of Uncertainty | Symbol $(x_i)$ | Type | Uncertainty Value (˚C) | Probability Distribution | Coverage Factor | Standard Uncertainty $u(x_i)$ (°C) | Sensitivity Coefficient $c_i$ | $|c_i| \times u(x_i)$ (˚C) | Degrees of freedom $(\nu)$ |
|---|---|---|---|---|---|---|---|---|---|
| Repeatability of thermocouple reading | $u(t_{rdg})$ | A | 0.033 | - | - | 0.033 | 1 | 0.033 | 9 |
| Thermocouple correction | $u(\Delta t_{tc})$ | B | 1.0 | Normal | 2.000 | 0.500 | 1 | 0.500 | $\infty$ |
| Thermocouple immersion correction | $u(\Delta t_{imm})$ | B | 0.1 | Rectangular | 1.732 | 0.058 | 1 | 0.058 | $\infty$ |
| Thermocouple drift correction | $u(\Delta t_{drift})$ | B | 0.2 | Rectangular | 1.732 | 0.115 | 1 | 0.115 | $\infty$ |
| Digital thermometer deviation correction | $u(\Delta t_{ind})$ | B | 0.6 | Rectangular | 1.732 | 0.346 | 1 | 0.346 | $\infty$ |
| Digital thermometer resolution correction | $u(\Delta t_{res})$ | B | 0.05 | Rectangular | 1.732 | 0.029 | 1 | 0.029 | $\infty$ |

**EXAMPLE 2 : TEMPERATURE MEASUREMENT USING THERMOCOUPLE**

## 6.0    COMBINED STANDARD UNCERTAINTY

The combined standard uncertainty $u_c(t_x)$ of the measured temperature is :

$$u_c(t_x) = \sqrt{0.033^2 + 0.500^2 + 0.058^2 + 0.115^2 + 0.346^2 + 0.029^2}$$
$$= 0.623°\,C$$

## 7.0    EFFECTIVE DEGREES OF FREEDOM

Effective degrees of freedom of the combined standard uncertainty is :

$$\nu_{eff} = \frac{0.623^4}{\dfrac{0.033^4}{9} + \dfrac{0.500^4}{\infty} + \dfrac{0.058^4}{\infty} + \dfrac{0.115^4}{\infty} + \dfrac{0.346^4}{\infty} + \dfrac{0.029^4}{\infty}}$$
$$\approx \infty$$

## 8.0    EXPANDED UNCERTAINTY

The effective degrees of freedom for the combined standard uncertainty $u_c(t_x)$ was assumed to be infinity. This gives a coverage factor $k$ equals to 2 at a level of confidence of approximately 95 %. Therefore, the expanded uncertainty is :

$$U = k\,u_c(t_x)$$
$$= 2 \times 0.623\,°C$$
$$= 1.3\ °C$$

## 9.0    REPORTING OF RESULTS

The temperature of the chamber after taking into consideration the correction of the thermocouple is 400.5 °C. The measurement uncertainty is ± 1.3 °C, estimated at a level of confidence of approximately 95 % with a coverage factor $k$ equals to 2.

**EXAMPLE 3 : CALIBRATION OF PLATINUM RESISTANCE OF THERMOMETER (PRT)**

## 1.0    INTRODUCTION

A semi-standard 100 ohm platinum resistance thermometer (PRT) was calibrated by comparison against a 25 ohm Standard Platinum Resistance Thermometer (SPRT) traceable to the International Temperature Scale 1990 (ITS-90). The calibration was carried out at nominal temperatures of 0 °C, 50 °C, 100 °C, 190 °C, 230 °C, 315 °C and 420 °C using three different types of fluid circulating baths. The schematic diagram of the PRT calibration is as shown in Figure 1.
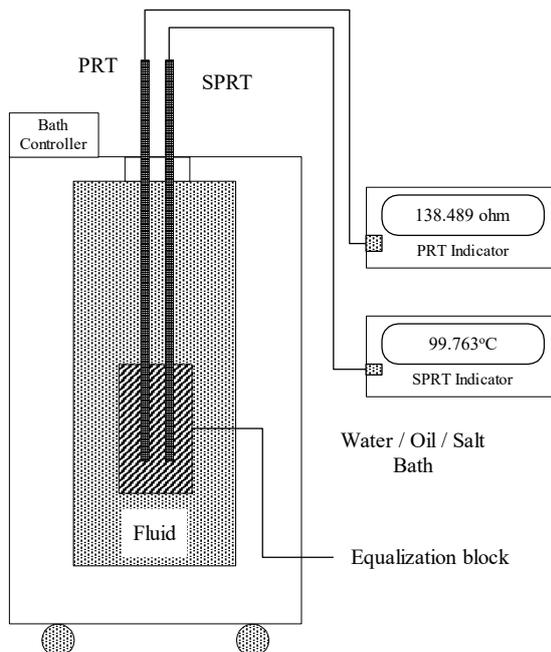


Figure 1:Schematic diagram for PRT calibration

The measurement data are given in Table 1. These data were collected in ascending order of temperatures in degree Celsius. The resistance of the PRT at 0 °C was measured twice, at the beginning and at the end of the calibration using an ice-pot. The temperatures of the baths were measured by the SPRT. The PRT resistance measurements were carried out at 1 mA current that did not include any equivalent resistance due to thermal emf in the PRT.

**Table 1**

| SPRT temperature [$t_{SPRT}$ :°C] | PRT resistance [$R_{PRT}$ :Ω] |
|---|---|
| 0.000 | 100.021 |
| 50.131 | 119.498 |
| 99.763 | 138.489 |
| 189.869 | 172.232 |
| 229.996 | 186.955 |
| 315.008 | 217.539 |
| 419.573 | 254.023 |
| 0.000 | 100.019 |

**EXAMPLE 3 : CALIBRATION OF PLATINUM RESISTANCE OF THERMOMETER (PRT)**

## 2.0    MATHEMATICAL MODEL

The uncertainty evaluation was carried out in two parts. Firstly, determination of the test temperature of which the PRT was calibrated and, secondly, determination of the resistance of the PRT at the test temperature.

The test temperature ($t_x$) was given by:

$$t_x \cong t_{SPRT} + c_{SPRT}\,\Delta r_{ind} + c_{SPRT}\,\Delta r_{res} + \Delta t_{rep} + \Delta t_{imm} + \Delta t_{heat} + \Delta t_{drift} + \Delta t_{bath} \;...(1)$$

The resistance ($R_x$) of the PRT at test temperature was given by:

$$R_x \cong R_{PRT} + \Delta R_{ind} + \Delta R_{res} + \Delta R_{rep} + \Delta R_{imm} + \Delta R_{hys} \qquad ...(2)$$

where

| | |
|---|---|
| $t_{SPRT}$ | : temperature of the SPRT. |
| $R_{PRT}$ | : measured resistance of the PRT at test temperature. |
| $\Delta r_{ind}$ , $\Delta R_{ind}$ | : resistance correction due to deviation of the indicator of the thermometer. |
| $\Delta r_{res}$ , $\Delta R_{res}$ | : resistance correction due to resolution of the indicator of the thermometer. |
| $\Delta t_{rep}/\Delta R_{rep}$ | : temperature/resistance correction due to repeatability of the SPRT and PRT. |
| $\Delta t_{imm}/\Delta R_{imm}$ | : temperature/resistance correction due to immersion error. |
| $\Delta t_{heat}$ | : temperature correction due to self-heating effects of the SPRT. |
| $\Delta t_{drift}$ | : temperature correction due to drift of the SPRT since last calibration. |
| $\Delta t_{bath}$ | : temperature correction due to non-uniformity of bath temperature. |
| $\Delta R_{hys}$ | : resistance correction due to PRT hysteresis. |
| $c_{SPRT}$ | : sensitivity coefficient of SPRT ($\partial T/\partial R$). |

## 3.0    UNCERTAINTY EQUATION

The combined standard uncertainty $u_c(t_x)$ of the test temperature was given by:

$$u_c(t_x) \cong \left[ \begin{array}{l} u^2(t_{SPRT}) + c_{SPRT}^2\,u^2(\Delta r_{ind}) + c_{SPRT}^2\,u^2(\Delta r_{res}) + u^2(\Delta t_{rep}) + u^2(\Delta t_{imm}) \\ + u^2(\Delta t_{heat}) + u^2(\Delta t_{drift}) + u^2(\Delta t_{bath}) \end{array} \right]^{1/2} \qquad ...(3)$$

**EXAMPLE 3 : CALIBRATION OF PLATINUM RESISTANCE OF THERMOMETER (PRT)**

The combined standard uncertainty $u_c(R_x)$ of the PRT resistance at test temperature was given by:

$$u_c(R_x) \cong \left[ \frac{u_c^2(t_x)}{c_{PRT}^2} + u^2(\Delta R_{ind}) + u^2(\Delta R_{res}) + u^2(\Delta R_{rep}) + u^2(\Delta R_{imm}) + u^2(\Delta R_{hys}) \right]^{\frac{1}{2}} \qquad ...(4)$$

where

| | |
|---|---|
| $u(t_{SPRT})$ | : standard uncertainty of the SPRT. |
| $u(R_{PRT})$ | : standard uncertainty of measured resistance of the PRT at test temperature was given by the combined standard uncertainty $(u_c(t_x))$ of the test temperature evaluated in Equation 3 divided by the sensitivity coefficient of PRT ($c_{PRT}$). ($= u_c(t_x)/ c_{PRT}$). |
| $u(\Delta r_{ind})$, $u(\Delta R_{ind})$ | : standard uncertainty of the indicator correction. |
| $u(\Delta r_{res})$, $u(\Delta R_{res})$ | : standard uncertainty of the indicator's resolution correction. |
| $u(\Delta t_{rep})$, $u(\Delta R_{rep})$ | : standard uncertainty of the SPRT and PRT repeatability. |
| $u(\Delta t_{imm})$, $u(\Delta R_{imm})$ | : standard uncertainty of the SPRT and PRT immersion correction. |
| $u(\Delta t_{heat})$ | : standard uncertainty of the SPRT self-heating effects correction. |
| $u(\Delta t_{drift})$ | : standard uncertainty of the SPRT drift correction. |
| $u(\Delta t_{bath})$ | : standard uncertainty of the temperature correction for bath non-uniformity bath. |
| $u(\Delta R_{hys})$ | : standard uncertainty of the PRT hysteresis correction. |
| $c_{PRT}$ | : sensitivity coefficient of PRT ($\partial T/\partial R$). |

The typical sensitivity coefficient ($c_{SPRT}$) of a 25 ohm SPRT is 10.0 °C/Ω, whereas the typical sensitivity coefficient ($c_{PRT}$) of a 100 Ω PRT is 2.5 °C/Ω, and the inverse of $c_{PRT}$ ($1/c_{PRT}$), as required in Equation 4 is 0.4 Ω/°C.

## 4.0    EVALUATION OF UNCERTAINTY

No correction was applied to any of the correction terms stated in Equation 1 and 2.

| Symbol | Information | Uncertainty value |
|---|---|---|
| $t_{SPRT}$ | The SPRT calibration certificate stated that expanded uncertainty ($U$) was ± 0.004 °C with a coverage factor $k = 2$. | ± 0.004 °C |
| $\Delta r_{ind}$ / $\Delta R_{ind}$ | The specifications of the SPRT and PRT indicators were ± 0.0002 Ω and ± 0.008 Ω respectively. | ± 0.0002 Ω [1] / ± 0.008 Ω |
| $\Delta r_{res}$ / $\Delta R_{res}$ | No correction was applied due to the resolution of the indicator. The uncertainty value associated with this correction was taken to be half of the least significant digit of the indicator.<br><br>SPRT indicator's resolution: 0.0001 Ω,<br>PRT indicator's resolution: 0.001 Ω | ± 0.00005 Ω / ± 0.0005 Ω |

**EXAMPLE 3 : CALIBRATION OF PLATINUM RESISTANCE OF THERMOMETER (PRT)**

| Symbol | Information | Uncertainty value |
|---|---|---|
| $\Delta t_{rep}$ / $\Delta R_{rep}$ | The maximum standard errors of the average readings of the SPRT and PRT were $\pm$ 0.002 °C ($\pm$ 0.0002 $\Omega$) and $\pm$ 0.001 $\Omega$ respectively. | $\pm$ 0.002 °C [2] / $\pm$ 0.001 $\Omega$ |
| $\Delta t_{imm}$ / $\Delta R_{imm}$ | The immersion depth during the calibration was 300 mm. Both the SPRT and PRT were sufficiently immersed into the circulating baths such that the heat leakage from or to the sensing element was negligible. An uncertainty limit of $\pm$ 0.001 °C was estimated for the SPRT and $\pm$ 0.001 $\Omega$ for the PRT. | $\pm$ 0.001 °C/ $\pm$ 0.001 $\Omega$ |
| $\Delta t_{heat}$ | The SPRT was calibrated at 1 mA measuring current. No correction was applied to the SPRT due to the self-heating effects contributed by the 1 mA measuring current at the time of usage. An uncertainty limit of $\pm$ 0.001 °C was estimated. | $\pm$ 0.001 °C |
| $\Delta t_{drift}$ | No correction was made for the SPRT drift since the last calibration. The uncertainty limit of this correction was $\pm$ 0.002 °C based on the previous calibrations. | $\pm$ 0.002 °C |
| $\Delta t_{bath}$ | No correction was applied for the bath non uniformity inside the calibration baths. The uncertainty limit of this correction value was found to be within $\pm$ 0.005 °C. | $\pm$ 0.005 °C |
| $\Delta R_{hys}$ | No correction was applied for the hysteresis effect. The uncertainty of the hysteresis correction was estimated to be the change in the PRT resistance at 0.0 °C, which was 0.002 $\Omega$. | $\pm$ 0.002 $\Omega$ |

**Remarks**

[1] The calibration certificates of the indicators results showed that the errors of the indicators and calibration uncertainties were within the specifications of the indicators. No correction was applied to the indicator resistance readings. Drift and other influences such as environmental effects were considered negligible, therefore only the specifications of the indicators were included.

[2] The SPRT and PRT readings were taken simultaneously in order to minimise the error caused by the temporal fluctuations of the bath. After the bath had stabilised, thirty SPRT and PRT readings at a nine seconds interval were observed.

**EXAMPLE 3 : CALIBRATION OF PLATINUM RESISTANCE OF THERMOMETER (PRT)**

## 5.0 UNCERTAINTY BUDGET TABLE

Uncertainty budget table for the test temperature ($t_x$) was tabulated below.

| Source of Uncertainty | Symbol $x_i$ | Type | Uncertainty Value | Probability Distribution | Coverage Factor | Standard uncertainty $u(x_i)$ | Sensitivity Coefficient $c_i$ | $|c_i| \times u(x_i)$ (°C) | Degrees of freedom $\nu$ |
|---|---|---|---|---|---|---|---|---|---|
| SPRT | $t_{SPRT}$ | B | 0.004 °C | Normal | 2 | 0.0020 | 1 | 0.0020 | ∞ |
| SPRT indicator specifications | $\Delta r_{ind}$ | B | 0.0002 Ω | Rectangular | 1.732 | 0.00012 | 10 | 0.0012 | ∞ |
| SPRT indicator resolution | $\Delta r_{res}$ | B | 0.00005 Ω | Rectangular | 1.732 | 0.000029 | 10 | 0.00029 | ∞ |
| SPRT repeatability | $\Delta t_{rep}$ | A | 0.002 °C | - | - | 0.0020 | 1 | 0.0020 | 29 |
| SPRT immersion | $\Delta t_{imm}$ | B | 0.001 °C | Rectangular | 1.732 | 0.00058 | 1 | 0.00058 | ∞ |
| SPRT self-heating | $\Delta t_{heat}$ | B | 0.001 °C | Rectangular | 1.732 | 0.00058 | 1 | 0.00058 | ∞ |
| SPRT drift | $\Delta t_{drift}$ | B | 0.002 °C | Rectangular | 1.732 | 0.0012 | 1 | 0.0012 | ∞ |
| Bath non-uniformity | $\Delta t_{bath}$ | B | 0.005 °C | Rectangular | 1.732 | 0.0029 | 1 | 0.0029 | ∞ |

Uncertainty budget table for the PRT resistance ($R_x$) at test temperature ($t_x$) was tabulated below.

| Source of Uncertainty | Symbol $x_i$ | Type | Uncertainty Value | Probability Distribution | Coverage factor | Standard uncertainty $u(x_i)$ | Sensitivity Coefficient $c_i$ | $|c_i| \times u(x_i)$ (Ω) | Degrees of freedom $\nu$ |
|---|---|---|---|---|---|---|---|---|---|
| Test temperature | $t_x$ | B | 0.0045 °C | Normal | 1 | 0.0045 | 0.4 | 0.0018 | 700 |
| PRT indicator specifications | $\Delta R_{ind}$ | B | 0.008 Ω | Rectangular | 1.732 | 0.0046 | 1 | 0.0046 | ∞ |
| PRT indicator resolution | $\Delta R_{res}$ | B | 0.0005 Ω | Rectangular | 1.732 | 0.00029 | 1 | 0.00029 | ∞ |
| PRT repeatability | $\Delta R_{rep}$ | A | 0.001 Ω | - | - | 0.0010 | 1 | 0.0010 | 29 |
| PRT immersion | $\Delta R_{imm}$ | B | 0.001 Ω | Rectangular | 1.732 | 0.00058 | 1 | 0.00058 | ∞ |
| PRT hysteresis | $\Delta R_{hys}$ | B | 0.002 Ω | Rectangular | 1.732 | 0.0012 | 1 | 0.0012 | ∞ |

**EXAMPLE 3 : CALIBRATION OF PLATINUM RESISTANCE OF THERMOMETER (PRT)**

### 6.0 COMBINED STANDARD UNCERTAINTY

The combined standard uncertainty $u_c(t_x)$ of the test temperature ($t_x$) was given below.

$$u_c(t_x) = \left[0.0020^2 + 0.0012^2 + 0.00029^2 + 0.0020^2 + 0.00058^2 + 0.00058^2 + 0.0012^2 + 0.0029^2\right]^{\frac{1}{2}}$$
$$= 0.0045 \;^{\circ}\text{C}$$

The combined standard uncertainty $u_c(R_x)$ of the PRT resistance was given below.

$$u_c(R_x) = \left[0.0018^2 + 0.0046^2 + 0.00029^2 + 0.0010^2 + 0.00058^2 + 0.0012^2\right]^{\frac{1}{2}}$$
$$= 0.0053 \; \Omega$$

### 7.0 EFFECTIVE DEGREES OF FREEDOM

Effective degrees of freedom of the combined standard uncertainty $u_c(t_x)$ was :

$$\nu_{eff} = \frac{0.0045^4}{\dfrac{0.0020^4}{\infty} + \dfrac{0.0012^4}{\infty} + \dfrac{0.00029^4}{\infty} + \dfrac{0.0020^4}{29} + \dfrac{0.00058^4}{\infty} + \dfrac{0.00058^4}{\infty} + \dfrac{0.0012^4}{\infty} + \dfrac{0.0029^4}{\infty}}$$
$$\approx 700$$

Effective degrees of freedom of the combined standard uncertainty $u_c(R_x)$ was :

$$\nu_{eff} = \frac{0.0053^4}{\dfrac{0.0018^4}{700} + \dfrac{0.0046^4}{\infty} + \dfrac{0.00029^4}{\infty} + \dfrac{0.0010^4}{29} + \dfrac{0.00058^4}{\infty} + \dfrac{0.0012^4}{\infty}}$$
$$\approx 15000$$

### 8.0 EXPANDED UNCERTAINTY

The effective degrees of freedom for the combined standard uncertainty $u_c(R_x)$ was taken to be infinity. At a level of confidence of approximately 95 %, a coverage factor $k$ of 2 was used. The expanded uncertainty,

$$U = k\,u_c(R_x)$$
$$= 2 \text{ x } 0.0053 \; \Omega$$
$$= 0.011 \; \Omega$$

**EXAMPLE 3 : CALIBRATION OF PLATINUM RESISTANCE OF THERMOMETER (PRT)**

## 9.0 REPORTING OF RESULTS

| Test temperature [$t_x$ : °C] | PRT resistance [$R_x$ : Ω] |
|---|---|
| 0.000 | 100.020 |
| 50.131 | 119.498 |
| 99.763 | 138.489 |
| 189.869 | 172.232 |
| 229.996 | 186.955 |
| 315.008 | 217.539 |
| 419.573 | 254.023 |

The expanded uncertainty of the PRT resistance given at the test temperatures was ± 0.011 Ω, estimated at a level of confidence of approximately 95 % with a coverage factor of $k = 2$.

**Remark :**

The following cubic equation was used to interpolate the PRT resistance from 0 °C to 420 °C, where W(t) = R(t) / R (0 °C). The coefficients a, b and c were determined by fitting the calibration results using the method of least squares.

$$W(t) = 1 + a\left(\frac{t}{100}\right) + b\left(\frac{t}{100}\right)^2 + c\left(\frac{t}{100}\right)^3$$

a = 3.914069 x $10^{-1}$, b = -5.918921 x $10^{-3}$ , c = 2.281030 x $10^{-5}$

## EXAMPLE 4 : DC CURRENT MEASUREMENT


## 1.0     INTRODUCTION

A current of 10A is measured by using a current shunt and a voltmeter. The specifications of the instruments used are as follows:
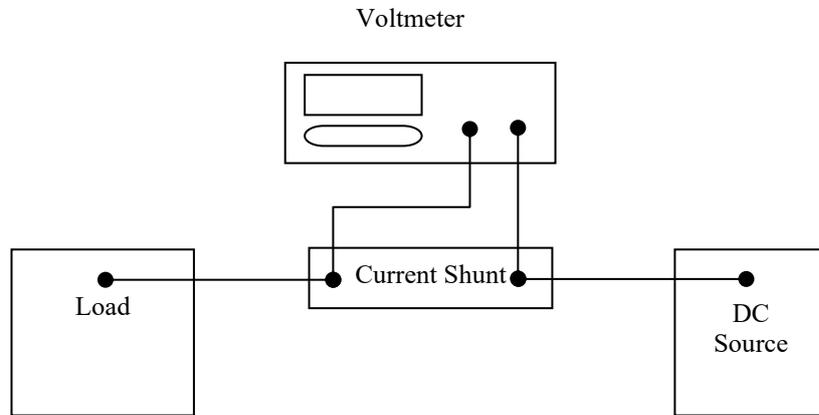


Figure 1: Measurement set-up

1.1     Current Shunt Specifications
- • Current : 10 A
- • Resistance : 0.01 $\Omega$
- • The current shunt was calibrated recently.  From the calibration report, we have the following:
  - Resistance R=0.010088 $\Omega$ measured at 10 A, 23 $^\circ$C
  - Relative expanded uncertainty : ±0.08 % at coverage factor $k = 2$
  - Temperature coefficient in the range 15 $^\circ$C to 30 $^\circ$C: 60 ppm / K
  - The relative expanded uncertainties can be converted into expanded uncertainty by multiplying by R.
- • The uncertainty due to drift of resistance value was found negligible.

1.2     Voltmeter Specifications
- • All uncertainty[1] specifications apply for one year after calibration when operated in a temperature of 15 $^\circ$C to 40 $^\circ$C and a relative humidity of up to 80 %.
- • DC voltage function specification

| Range | Full scale | Uncertainty[1] ± (% of reading + number of counts) |
|---|---|---|
| 200 mV | 199.99 mV | 0.030 + 2 |

Note: [1] Traditionally the term accuracy is used

**EXAMPLE 4 : DC CURRENT MEASUREMENT**

1.3     Measurement Record

Room Temperature : $(23 \pm 5)$ $^{\circ}$C, Humidity : $(55 \pm 10)$ % Relative Humidity

| Reading no. | Voltage (mV) |
|:---:|:---:|
| 1 | 100.68 |
| 2 | 100.83 |
| 3 | 100.79 |
| 4 | 100.64 |
| 5 | 100.63 |
| 6 | 100.94 |
| 7 | 100.60 |
| 8 | 100.68 |
| 9 | 100.76 |
| 10 | 100.65 |

**2.0     MATHEMATICAL MODEL**

The current is a function of the voltage and resistance. From Ohm's law,

$$I = f(x_1, x_2)$$
$$= f(V, R)$$
$$= \frac{V}{R} \qquad \qquad \text{...(1)}$$

where
$I$ is the current, $V$ is the voltage, and $R$ is the resistance.

The best estimate of the voltage $V$ is the mean voltage $\overline{V}$ of a series of voltage measurement.

**EXAMPLE 4 : DC CURRENT MEASUREMENT**

### 3.0   UNCERTAINTY EQUATION

For uncorrelated input quantities, the combined standard uncertainty $u_c(l)$ of the current measured is :

$$u_c^2(I) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) \qquad \ldots(2)$$

Major sources of uncertainties in the measurement include:

(1)   Uncertainties due to the measured voltage which consists of the random variability of the measured voltage and the voltmeter uncertainty. The effect of the room temperature on voltmeter has already been taken into account in the error limits.

(2)   Uncertainties due to the current shunt resistance which consist of the calibrated resistance value and the resistance change due to temperature effect.  The uncertainty due to drift of resistance value is negligible.

Let,

$u_1(V)$ : Standard uncertainty of voltage due to random effect.

$u_2(V)$ : Standard uncertainty of voltage due to voltmeter uncertainty.

$u_3(R)$ : Standard uncertainty of current shunt calibrated resistance value.

$u_4(R)$ : Standard uncertainty of current shunt resistance due to temperature effect.

Then,

$$u_c^2(I) = \left( \frac{\partial I}{\partial V} \right)^2 u_1^2(V) + \left( \frac{\partial I}{\partial V} \right)^2 u_2^2(V) + \left( \frac{\partial I}{\partial R} \right)^2 u_3^2(R) + \left( \frac{\partial I}{\partial R} \right)^2 u_4^2(R)$$

$$= \left[ c_1 \cdot u_1(V) \right]^2 + \left[ c_1 \cdot u_2(V) \right]^2 + \left[ c_2 \cdot u_3(R) \right]^2 + \left[ c_2 \cdot u_4(R) \right]^2$$

$$= (c_1)^2 \left[ u_1^2(V) + u_2^2(V) \right] + (c_2)^2 \left[ u_3^2(R) + u_4^2(R) \right] \qquad \ldots(3)$$

and the sensitivity coefficients are:

$$c_1 \equiv \frac{\partial I}{\partial V} = \frac{1}{R} \qquad \ldots(4)$$

$$c_2 \equiv \frac{\partial I}{\partial R} = -\frac{V}{R^2} \qquad \ldots(5)$$

**EXAMPLE 4 : DC CURRENT MEASUREMENT**


**4.0     ANALYSIS OF UNCERTAINTY COMPONENTS**

4.1     Type A Evaluation

4.1.1   Voltage
The data are tabulated in Table 1 for the calculation of mean voltage and standard deviation of the voltage.

Table 1: Data for calculation of mean and standard deviation of voltage

| No. | $V_i$ (mV) | $\left(V_i - \overline{V}\right) \times 10^{-2}$ | $\left(V_i - \overline{V}\right)^2 \times 10^{-4}$ |
|---|---|---|---|
| 1 | 100.68 | -4 | 16 |
| 2 | 100.83 | 11 | 121 |
| 3 | 100.79 | 7 | 49 |
| 4 | 100.64 | -8 | 64 |
| 5 | 100.63 | -9 | 81 |
| 6 | 100.94 | 22 | 484 |
| 7 | 100.60 | -12 | 144 |
| 8 | 100.68 | -4 | 16 |
| 9 | 100.76 | 4 | 16 |
| 10 | 100.65 | -7 | 49 |
| Total | 1007.2 | 0 | 1040 |

Mean voltage:

$$\overline{V} = \frac{\sum\limits_{i=1}^{n} V_i}{n}$$
$$= \frac{1007.20}{10}$$
$$= 100.72 \text{ mV} \qquad \qquad ...(6)$$

The best estimate of the voltage is therefore:

$$V = \overline{V} = 100.72 \text{ mV} \qquad \qquad ...(7)$$

**EXAMPLE 4 : DC CURRENT MEASUREMENT**

Standard deviation:

$$s(V_i) = \sqrt{\left(\frac{1}{n-1}\right)\sum_{i=1}^{n}\left(V_i - \overline{V}\right)^2}$$
$$= \sqrt{\left(\frac{1}{9}\right)\left(1040 \times 10^{-4}\right)}$$
$$= 10.75 \times 10^{-2} \text{ mV} \qquad \qquad ...(8)$$

Standard deviation of the mean:

$$s(\overline{V}) = \frac{s(V_i)}{\sqrt{n}}$$
$$= \frac{10.75 \times 10^{-2}}{\sqrt{10}}$$
$$= 3.40 \times 10^{-2} \text{ mV} \qquad \qquad ...(9)$$

Standard uncertainty of voltage due to random effect is:

$$u_1(V) = 3.40 \times 10^{-2} \text{ mV} \qquad \qquad ...(10)$$

Degrees of freedom:

$$v_1 = n - 1$$
$$= 10 - 1$$
$$= 9 \qquad \qquad ...(11)$$

4.2     Type B Evaluation

4.2.1   Uncertainty of Voltmeter
        From specification, the uncertainty of the voltmeter is:

$$a_2 = \left(0.030 \text{ \% of reading} + 2 \text{ counts}\right)$$
$$= \left(0.030 \text{ \%} \times 100.72 \text{ mV} + 0.02 \text{ mV}\right)$$
$$= 5.02 \times 10^{-2} \text{ mV} \qquad \qquad ...(12)$$

**EXAMPLE 4 : DC CURRENT MEASUREMENT**

Assumed rectangular distribution, standard uncertainty of voltage due to voltmeter uncertainty is:

$$u_2(V) = \frac{a_2}{\sqrt{3}}$$
$$= \frac{5.02 \times 10^{-2}}{\sqrt{3}}$$
$$= 2.90 \times 10^{-2} \text{ mV} \qquad \qquad ...(13)$$

Degrees of freedom:

$$v_2 = \infty \qquad \qquad ...(14)$$

4.2.2   Current Shunt Resistance

From calibration report, the uncertainty of the shunt resistance is:

$$a_3 = 8 \times 10^{-4} \times R$$
$$= 8 \times 10^{-4} \times 0.010088$$
$$= 8.07 \times 10^{-6} \text{ } \Omega \qquad \qquad ...(15)$$

Given coverage factor $k = 2$, the standard uncertainty of current shunt resistance is:

$$u_3(R) = \frac{a_3}{k}$$
$$= \frac{8.07 \times 10^{-6}}{2}$$
$$= 4.04 \times 10^{-6} \text{ } \Omega \qquad \qquad ...(16)$$

Degrees of freedom:

$$v_3 = \infty \qquad \qquad ...(17)$$

4.2.3   Temperature Effect on Current Shunt ($23 \pm 5$ ºC)

From the calibration report, the uncertainty of the shunt resistance due to the effect of room temperature $\Delta t = 5$ °C is:

$$a_4 = 60 \times 10^{-6} \times \Delta t \times R$$
$$= 60 \times 10^{-6} \times 5 \times 0.010088$$
$$= 3.03 \times 10^{-6} \text{ } \Omega \qquad \qquad ...(18)$$

**EXAMPLE 4 : DC CURRENT MEASUREMENT**

Assumed rectangular distribution, the standard uncertainty of resistance due to the temperature effect is:

$$u_4(R) = \frac{a_4}{\sqrt{3}}$$

$$= \frac{3.03 \times 10^{-6}}{\sqrt{3}}$$

$$= 1.75 \times 10^{-6} \ \Omega \qquad \qquad \text{...(19)}$$

Degrees of freedom:

$$v_4 = \infty \qquad \qquad \text{...(20)}$$

## 5.0    CALCULATION OF CURRENT

$$I = \frac{V}{R}$$

$$= \frac{100.72 \times 10^{-3}}{0.010088}$$

$$= 9.984 \ A \qquad \qquad \text{...(21)}$$

## 6.0    SENSITIVITY COEFFICIENTS

$$c_1 = \frac{1}{R}$$

$$= \frac{1}{0.010088}$$

$$= 99.128 \ \Omega^{-1} \qquad \qquad \text{...(22)}$$

$$c_2 = -\frac{V}{R^2}$$

$$= -\frac{100.72 \times 10^{-3}}{0.001088}$$

$$= -989.70 \ V/\Omega^2 \qquad \qquad \text{...(23)}$$

# EXAMPLE 4 : DC CURRENT MEASUREMENT

## 7.0     UNCERTAINTY BUDGET

| Source of Uncertainty | Symbol $x_I$ | Type | Uncertainty Value | Probability Distribution | Coverage Factor | Standard Uncertainty $u(x_i)$ | Sensitivity Coefficient $c_i$ | $\lvert c_i \rvert$ x $u(x_i)$ (A) | Degrees of Freedom $\nu$ |
|---|---|---|---|---|---|---|---|---|---|
| Repeatability | $u_1(V)$ | A | $3.40 \times 10^{-2}$ mV | - | - | $3.40 \times 10^{-2}$ mV | 99.128 $\Omega^1$ | $3.37 \times 10^{-3}$ | 9 |
| Voltmeter | $u_2(V)$ | B | $5.02 \times 10^{-2}$ mV | Rectangular | $\sqrt{3}$ | $2.90 \times 10^{-2}$ mV | 99.128 $\Omega^{-1}$ | $2.87 \times 10^{-3}$ | $\infty$ |
| Shunt Resistance | $u_3(R)$ | B | $8.07 \times 10^{-6}$ $\Omega$ | Normal | 2 | $4.04 \times 10^{-6}$ $\Omega$ | 989.7 $V/\Omega^2$ | $3.99 \times 10^{-3}$ | $\infty$ |
| Shunt Temperature Effect | $u_4(R)$ | B | $3.03 \times 10^{-6}$ $\Omega$ | Rectangular | $\sqrt{3}$ | $1.75 \times 10^{-6}$ $\Omega$ | 989.7 $V\Omega^2$ | $1.73 \times 10^{-3}$ | $\infty$ |

## 8.0     COMBINED STANDARD UNCERTAINTY $u_c(y)$

For uncorrelated input quantities:

$$u_c^2(I) = c_1^2 \left[ u_1^2(V) + u_2^2(V) \right] + c_2^2 \left[ u_3^2(R) + u_4^2(R) \right] \qquad \text{...(24)}$$

Then,

$$u_c^2(I) = (99.128)^2 \left[ \left( 3.40 \times 10^{-2} \right)^2 + \left( 2.90 \times 10^{-2} \right)^2 \right] \left( 10^{-3} \right)^2$$

$$+ \left( -989.70 \right)^2 \left[ \left( 4.04 \times 10^{-6} \right)^2 + \left( 1.75 \times 10^{-6} \right)^2 \right] \qquad \text{...(25)}$$

Hence,

$$u_c(I) = \sqrt{3.86 \times 10^{-5}}$$
$$= 6.2 \times 10^{-3} \ \text{A} \qquad \text{...(26)}$$

## 9.0     EFFECTIVE DEGREES OF FREEDOM $\nu_{eff}$

$$\nu_{eff} = \frac{u_c^4(I)}{\dfrac{\left[ c_1 u_1(V) \right]^4}{\nu_1} + \dfrac{\left[ c_1 u_2(V) \right]^4}{\nu_2} + \dfrac{\left[ c_1 u_3(V) \right]^4}{\nu_3} + \dfrac{\left[ c_1 u_4(V) \right]^4}{\nu_4}}$$

$$= \frac{\left( 6.2 \times 10^{-3} \right)^4}{\dfrac{\left( 99.128 \times 3.40 \times 10^{-5} \right)^4}{9} + \dfrac{\left( 99.128 \times 2.90 \times 10^{-5} \right)^4}{\infty} + \dfrac{\left( -989.70 \times 4.04 \times 10^{-6} \right)^4}{\infty} + \dfrac{\left( -989.70 \times 1.75 \times 10^{-6} \right)^4}{\infty}}$$

$$\approx 103 \qquad \text{...(27)}$$

**EXAMPLE 4 : DC CURRENT MEASUREMENT**

## 10.0    EXPANDED UNCERTAINTY *U*

For effective degrees of freedom $v_{eff} > 100$, $k = 2$ can be used at approximately 95 % level of confidence. Therefore, the expanded uncertainty $U$ is :

$$
\begin{aligned}
U &= k\, u_c(I) \\
&= 2 \times 6.2 \times 10^{-2} \\
&= 0.012 \ \text{A}
\end{aligned}
\qquad \text{...(28)}
$$

## 11.0    REPORTING OF RESULTS

$$
I = (9.984 \pm 0.012) \ \text{A} \qquad \text{...(29)}
$$

The measured result is 9.984 A.  The reported expanded uncertainty is $\pm$ 0.012 A with a coverage factor of $k = 2$, assuming a normal distribution at a level of confidence of approximately 95 %.

## 12.0    REMARKS

12.1    Alternative method to calculate combined standard uncertainty $u_c$ by relative combined method. Since the mathematical model is of the form

$$
Y = c\, X_1^{P_1}\, X_2^{P_2}\, \cdots X_N^{P_N} \qquad \text{...(30)}
$$

The combined standard uncertainty $u_c(y)$ can be expressed as an estimated relative combined variance:

$$
\left( \frac{u_c(y)}{y} \right)^2 = \sum_{i=1}^{N} \left( \frac{u(x_i)}{x_i} \right)^2 \qquad \text{...(31)}
$$

Therefore,

$$
\begin{aligned}
\left( \frac{u_c(I)}{I} \right)^2 &= \left( \frac{u_1(V)}{V} \right)^2 + \left( \frac{u_2(V)}{V} \right)^2 + \left( \frac{u_3(R)}{R} \right)^2 + \left( \frac{u_4(R)}{R} \right)^2 \\
&= \left( \frac{3.40 \times 10^{-2}}{100.72} \right)^2 + \left( \frac{2.90 \times 10^{-2}}{100.72} \right)^2 + \left( \frac{4.04 \times 10^{-6}}{0.010088} \right)^2 + \left( \frac{1.75 \times 10^{-6}}{0.010088} \right)^2 \\
&= 3.87 \times 10^{-7} \ per\ unit^2 \qquad \text{...(32)}
\end{aligned}
$$

**EXAMPLE 4 : DC CURRENT MEASUREMENT**

Then,

$$\left(\frac{u_c(I)}{I}\right) = 6.2 \text{ x} 10^{-4} \text{ (per unit)} \qquad \text{...(33)}$$

And

$$u_c(I) = I \text{ x} 6.2 \text{ x} 10^{-4}$$
$$= 9.984 \text{ x} 6.2 \text{ x} 10^{-4}$$
$$= 6.2 \text{ x} 10^{-3} \text{ A} \qquad \text{...(34)}$$

12.2    Alternative method to calculate effective degrees of freedom,

$$\nu_{eff} = \frac{\left[\dfrac{u_c(I)}{(I)}\right]^4}{\dfrac{\left[\dfrac{u_1(V)}{V}\right]^4}{\nu_1} + \dfrac{\left[\dfrac{u_2(V)}{V}\right]^4}{\nu_2} + \dfrac{\left[\dfrac{u_3(R)}{R}\right]^4}{\nu_3} + \dfrac{\left[\dfrac{u_4(R)}{R}\right]^4}{\nu_4}}$$

$$= \frac{\left[\dfrac{6.2 \text{ x} 10^{-3}}{9.984}\right]^4}{\dfrac{\left[\dfrac{3.40 \text{ x} 10^{-2}}{100.72}\right]^4}{9} + \dfrac{\left[\dfrac{2.90 \text{ x} 10^{-2}}{100.72}\right]^4}{\infty} + \dfrac{\left[\dfrac{4.04 \text{ x} 10^{-6}}{0.010088}\right]^4}{\infty} + \dfrac{\left[\dfrac{1.75 \text{ x} 10^{-6}}{0.010088}\right]^4}{\infty}}$$

$$\approx 103 \qquad \text{...(35)}$$

## EXAMPLE 5 : CALIBRATION OF DC 20 V RANGE OF A MULTIMETER

### 1.0     INTRODUCTION

The manufacturer's calibration procedure of a multi-meter required that a DC voltage of 10 V from a multifunction calibrator is applied to the multi-meter to verify its DC 20 V range accuracy. The specifications of the multifunction calibrator and digital multi-meter are as follows:
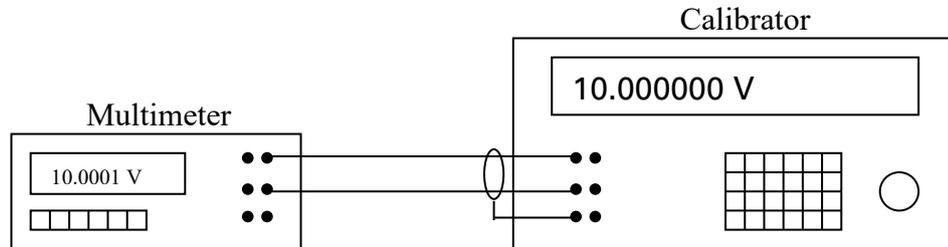


Figure 1 : Connection diagram

1.1     Multifunction Calibrator Specifications

The calibrator is calibrated every 90 days. The last calibration was carried out at a 23 $^{\circ}$C $\pm$ 1 $^{\circ}$C environment and the calibrator was verified to be within specifications.

DC Voltage Specifications

| Range | Resolution | Uncertainty at 99 % level of confidence $\pm$ 5 $^{\circ}$C from calibration temperature | | |
|---|---|---|---|---|
| | | 24 Hours | 90 Days | 1 Year |
| | | $\pm$ (ppm output + $\mu$V) | | |
| 20 V | 1 $\mu$V | 4 + 3 | 5 + 4 | 8 + 5 |

1.2     Digital Multimeter Specifications

DC Voltage range

| Range | Full Scale Display | Resolution | 1 Year Uncertainty[1] $\pm$ (% of reading + % of range) |
|---|---|---|---|
| 20 V | 19.9999 V | 100 $\mu$V | 0.0035 + 0.0025 |

Note : [1] Traditionally the term accuracy is used

It was given in the service manual that a DC voltage of 10 V is used to verify the multimeter's DC 20 V range. The verification specification for the multimeter DC 20 V range is:

Verification specification

| Range | Input Voltage | Limits |
|---|---|---|
| 20 V | 10 V | $\pm$ 0.0007 V |

# EXAMPLE 5 : CALIBRATION OF DC 20 V RANGE OF A MULTIMETER

1.3    Measurement Record

Temperature : $23\ ^{\circ}\text{C} \pm 2\ ^{\circ}\text{C}$

Only one reading was taken as the indicated voltage remain unchanged. Errors due to loading effect and connection are negligible.

| Applied Voltage | Indicated Voltage |
|-----------------|-------------------|
| 10.000000 V     | 10.0001 V         |

## 2.0    MATHEMATICAL MODEL

The multimeter was connected directly to the calibrator. The model of the process is a function of voltages :

$$V_{DMM} = f(V)$$
$$= V_{STD} + \Delta V_{DMM} \qquad \text{...(1)}$$

where

$V_{DMM}$ : Multimeter indicated voltage.

$V_{STD}$ : Applied voltage from the calibrator.

$\Delta V_{DMM}$ : Error of the multimeter.

## 3.0    UNCERTAINTY EQUATION

The input quantities are uncorrelated. The combined standard uncertainty $u_c(V)$ is :

$$u_c^2(V) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial V_1} \right)^2 u^2(V) \qquad \text{...(2)}$$

The voltage from the calibrator is directly applied to the multimeter. Since errors due to loading effect and connection were found negligible, the components of the total measurement uncertainty consist of :

$u_1(V)$ : Calibrator's applied voltage uncertainty.
$u_2(V)$ : Multimeter's random effect uncertainty.

The combined standard uncertainty becomes :

$$u_c^2(V) = \left( \frac{\partial V_{DMM}}{\partial V_{STD}} \right)^2 u_1^2(V) + \left( \frac{\partial V_{DMM}}{\partial \Delta V_{DMM}} \right)^2 u_2^2(V)$$
$$= (c_1)^2 u_1^2(V) + (c_2)^2 u_2^2(V) \qquad \text{...(3)}$$

**EXAMPLE 5 : CALIBRATION OF DC 20 V RANGE OF A MULTIMETER**

The sensitivity coefficients are :

$$c_1 \equiv \frac{\partial V_{DMM}}{\partial V_{STD}} \qquad\qquad ...(4)$$
$$= 1$$

$$c_2 \equiv \frac{\partial V_{DMM}}{\partial \Delta V_{DMM}} \qquad\qquad ...(5)$$
$$= 1$$

The combined standard uncertainty becomes :

$$u_c^2(V) = u_1^2(V) + u_2^2(V)$$

$$u_c(V) = \sqrt{u_1^2(V) + u_2^2(V)} \qquad\qquad ...(6)$$

## 4.0    ANALYSIS OF UNCERTAINTY COMPONENTS

4.1    Type A Evaluation

This is a case where the reference standard's accuracy is much better than the device under test's. The multimeter's reading may remain unchanged, or sometimes the multimeter has a $\pm 1$ count flickering due to the multimeter's digitizing process. In this case, the Type A evaluation of standard uncertainty can be assumed negligible and the repeatability uncertainty can be treated as Type B uncertainty using the resolution error of the multimeter.

4.2    Type B Evaluation

4.2.1    Calibrator

From the calibrator's specification, the uncertainty of the applied voltage is :

$$a_1 = 5 \text{ ppm of output} + 4 \text{ } \mu V \qquad\qquad ...(7)$$
$$= 5 \times 10^{-6} \times 10 \text{ V} + 4 \times 10^{-6} \text{ V}$$
$$= 54 \times 10^{-6} \text{ V}$$

Given level of confidence is 99 %. Assume normal distribution, coverage factor $k = 2.58$, the standard uncertainty of applied voltage is :

$$u_1(V) = \frac{a_1}{2.58} \qquad\qquad ...(8)$$
$$= \frac{54 \times 10^{-6}}{2.58}$$
$$= 20.9 \times 10^{-6} \text{ V}$$

Degrees of freedom : $\nu_1 = \infty$

**EXAMPLE 5 : CALIBRATION OF DC 20 V RANGE OF A MULTIMETER**

   4.2.2   Multimeter

From multimeter's specifications, the 20 V range resolution is 100 µV (i.e. 1 count). Since the reading was unchanged, assumed the limit is half a count.

$$a_2 = \frac{100 \ \mu V}{2} \qquad \qquad ...(9)$$
$$= 50 \times 10^{-6} \ V$$

Assumed rectangular distribution, the standard uncertainty due to the resolution uncertainty of the multimeter is :

$$u_2(V) = \frac{a_2}{\sqrt{3}} \qquad \qquad ...(10)$$
$$= \frac{50 \times 10^{-6}}{\sqrt{3}}$$
$$= 28.9 \times 10^{-6} \quad V$$

Degrees of freedom : $\nu_2 = \infty$

## 5.0    UNCERTAINTY BUDGET

| Sources of Uncertainty | Symbol $u_i$ | Type | Uncertainty Value | Probability Distribution | Coverage Factor $k$ | Standard Uncertainty $u(x_i)$ (V) | Sensitivity Coefficient $c_i$ | $|c_i| \times u(x_i)$ (V) | Degrees of Freedom $\nu$ |
|---|---|---|---|---|---|---|---|---|---|
| Calibrator | $u_1(V)$ | B | 54x10⁻⁶ V | Normal | 2.58 | 20.9x10⁻⁶ | 1 | 20.9x10⁻⁶ | ∞ |
| Resolution | $u_2(V)$ | B | 50x10⁻⁶ V | Rectangular | $\sqrt{3}$ | 28.9x10⁻⁶ | 1 | 28.9x10⁻⁶ | ∞ |

## 6.0    COMBINED STANDARD UNCERTAINTY $u_c(y)$

The combined standard uncertainty of the indicated voltage at the multimeter is :

$$u_c(V) = \sqrt{u_1^2(V) + u_2^2(V)} \qquad \qquad ...(11)$$

$$= \sqrt{(20.9 \times 10^{-6})^2 + (28.9 \times 10^{-6})^2}$$

$$= 35.7 \times 10^{-6} \ V$$

# EXAMPLE 5 : CALIBRATION OF DC 20 V RANGE OF A MULTIMETER

## 7.0    EFFECTIVE DEGREES OF FREEDOM $v_{eff}$

Since $v_1$ and $v_2$ are infinite, effective degrees of freedom $v_{eff} = \infty$

$$v_{eff} = \frac{u_c^4(V)}{\dfrac{[c_1 u_1(V)]^4}{v_1} + \dfrac{[c_2 u_2(V)]^4}{v_2}} \qquad \text{...(12)}$$

$$= \frac{\left(3.57 \times 10^{-6}\right)^4}{\dfrac{\left(1 \times 20.9 \times 10^{-6}\right)^4}{\infty} + \dfrac{\left(1 \times 28.9 \times 10^{-6}\right)^4}{\infty}}$$

$$\approx \infty$$

## 8.0    EXPANDED UNCERTAINTY, $U$

For effective degrees of freedom $v_{eff} = \infty$, $k = 2$ is used at approximately 95 % level of confidence. Therefore, the expanded uncertainty is

$$U = k\ u_c(V) \qquad \text{...(13)}$$
$$= 2 \times 35.7 \times 10^{-6}$$
$$= 71\ \mu V$$

## 9.0    REPORTING OF RESULTS

$$V_{DMM} = 10.0001\ V \pm 71\ \mu V \qquad \text{...(14)}$$

The measured result is 10.0001 V. The reported expanded uncertainty is ± 71 µV with a coverage factor of $k = 2$, assuming a normal distribution at a level of confidence of approximately 95 %.

**EXAMPLE 6 : HARMONIC MEASUREMENT**

**1.0      INTRODUCTION**

The measurement configuration is shown in Figure 1. An attenuator is used to improve the mismatch uncertainty.
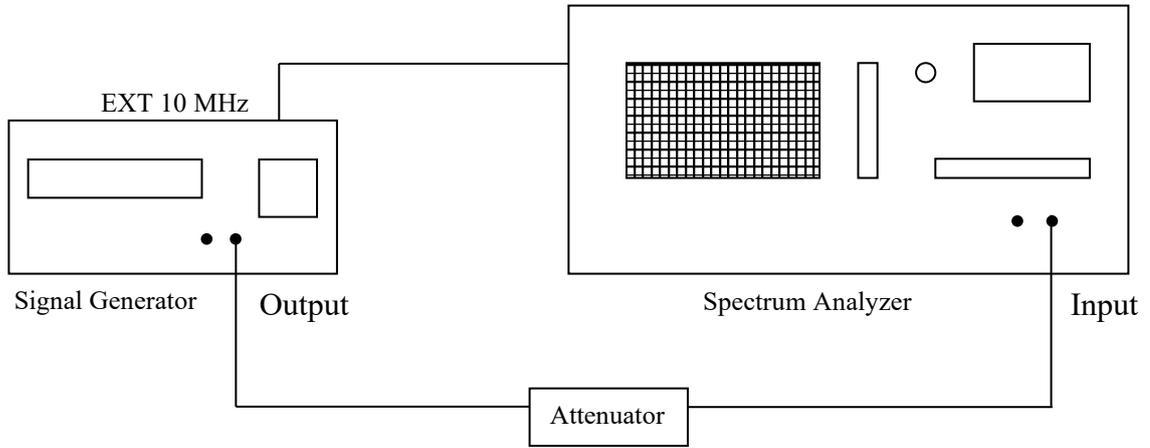


Figure 1: Measurement set-up

The following information is derived from the instrument's specification sheet.

Signal Generator (UUT) SWR          = 2.0
Spectrum Analyzer SWR               = 1.9
Spectrum analyzer instrument error  = 1.72 dB
Attenuation                         = 10 dB
Attenuator flatness                 = 1.2 dB
Cable flatness                      = 0.5 dB

The harmonic measurement data obtained from the spectrum analyzer are shown in Table 1.

Table 1

| No. of Readings | Data |
|-----------------|-------|
| 1 | 40.02 |
| 2 | 40.08 |
| 3 | 39.96 |
| 4 | 40.04 |
| 5 | 39.90 |

The mean of 5 harmonic measurements is 40 dBc.

**EXAMPLE 6 : HARMONIC MEASUREMENT**

## 1.1    The Measurand

The measurand, Harmonic  is the power difference between the CW carrier and a signal at a specified offset from the carrier. The offset is a multiple of the carrier frequency.

## 2.0    MEASUREMENT EQUATION

The measurement equation is:

$$Harmonic, Har = \frac{\left(\dfrac{Offset\_power}{Att_A}\right)}{\left(\dfrac{Carrier\_power}{Att_B}\right)} \times \left(1 + E_{flatness} + SA_{error}\right)$$

where,

| | |
|---|---|
| $Offset\_power$ | : measured level of the harmonic signal (mV). |
| $Carrier\_power$ | : measured level of the carrier Level (mW). |
| $Att_A$ | : path loss between the signal generator output  and the spectrum analyzer input while measuring the harmonic signal, including mismatch. |
| $Att_B$ | : path loss between the signal generator output  and the spectrum analyzer input while measuring the carrier (reference) signal, including mismatch. |
| $E_{flatness}$ | : variation in path loss of the cables, attenuator and adapters (not including SWR) when measuring the harmonic and reference signal as a fraction. |
| $SA_{error}$ | : error introduced by the spectrum analyzer as a fraction. |

**EXAMPLE 6 : HARMONIC MEASUREMENT**

**3.0    DERIVATION OF SENSITIVITY COEFFICIENT**

$$Harmonic, Har = f(Carrier\_power, Offset\_power, Att_A, Att_B, E_{flatness}, SA_{error})$$

$$c_1 = \frac{\partial Har}{\partial Carrier\_power} \qquad\qquad c_2 = \frac{\partial Har}{\partial Offset\_power}$$

$$c_3 = \frac{\partial Har}{\partial Att_A} \qquad\qquad c_4 = \frac{\partial Har}{\partial Att_B}$$

$$c_5 = \frac{\partial Har}{\partial E_{flatness}} \qquad\qquad c_6 = \frac{\partial Har}{\partial SA_{error}}$$

$c_1$ and $c_2$ do not need to be computed as the *Carrier_power* and *Offset_power* are simply the digital readings from the spectrum analyzer. The digital readings have no uncertainty associated with them. Instrumentation errors are contained in $SA_{error}$.

$$c_3 = \frac{-Offset\_power \times \left(1 + E_{flatness} + SA_{error}\right)}{Carrier\_power \times \dfrac{1}{Att_B} \times Att_A^2}$$

$$= \frac{-Offset\_power}{Carrier\_power \times Att_A}$$

Note :  The above is obtained by assuming $Att_A$ equal to $Att_B$. The second and third terms of the equation are negligible comparing to the first term and can be ignored.

Similarly,

$$c_4 = \frac{Offset\_power}{Carrier\_power \times Att_A}$$

$$c_5 = c_6$$
$$= \frac{Offset\_power \times Att_B}{Carrier\_power \times Att_A}$$
$$= \frac{Offset\_power}{Carrier\_power}$$

**EXAMPLE 6 : HARMONIC MEASUREMENT**

## 4.0    UNCERTAINTY EQUATION

The uncertainty equation is :

$$u(Har) = \sqrt{[c_3 \times u(Att_A)]^2 + [c_4 \times u(Att_B)]^2 + [c_5 \times u(E_{flatness})]^2 + [c_6 \times u(SA_{error})]^2}$$

$$= \frac{Offset\_power}{Carrier\_power} \times \sqrt{\left[\frac{u(Att_A)}{Att_A}\right]^2 + \left[\frac{u(Att_B)}{Att_A}\right]^2 + u^2\left(Cable_{flatness}\right) + u^2\left(SA_{error}\right)}$$

$$= \frac{Offset\_power}{Carrier\_power} \times \sqrt{2\left[\frac{u(Att_A)}{Att_A}\right]^2 + u^2\left(Cable_{flatness}\right) + u^2\left(SA_{error}\right)}$$

where
$u(x)$ is the uncertainty of variable $x$.

## 5.0    TYPE A UNCERTAINTY

Five readings were taken for the harmonic measurement. They are 40.02 dBc, 40.08 dBc, 39.96 dBc, 40.04 dBc and 39.90 dBc.

The mean is

$$\bar{x} = \frac{1}{5}\sum_{i=1}^{5} x_i$$
$$= 40 \text{ dBc}$$

The variance is

$$\frac{1}{5-1}\sum_{i=1}^{5}\left(x_i - \bar{x}\right)^2 = 0.005 \text{ dB}^2$$

The standard uncertainty,

$$u_A(Har) = \sqrt{\frac{0.005}{5}} \text{ dB}$$
$$= 0.0316 \text{ dB}$$

**EXAMPLE 6 : HARMONIC MEASUREMENT**

**6.0      TYPE B UNCERTAINTY DERIVATION**

6.1      $u(Att_A)$ and $u(Att_B)$ Derivation

The network consisting of the UUT, attenuator and spectrum analyzer can be represented in a signal flowgraph as below:
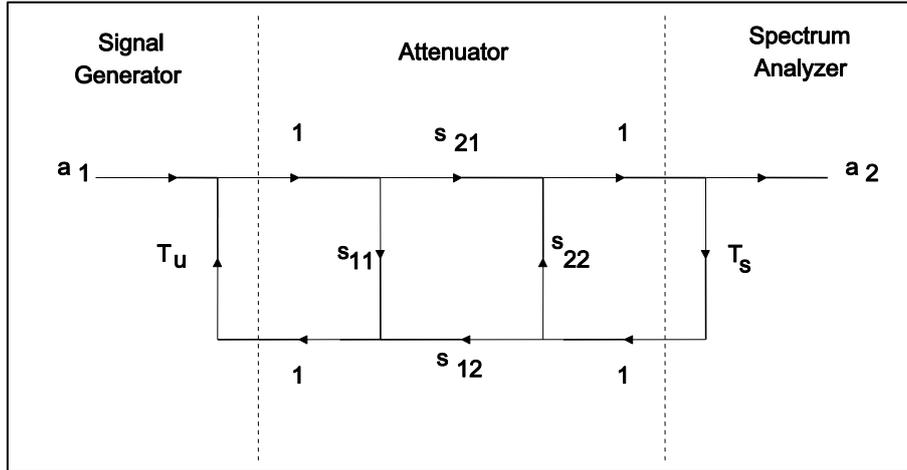


Figure 6.1:   Signal flow-graph representation of the measurement network

$T_u$ and $T_s$ are the reflection coefficient of the signal generator and spectrum analyzer respectively.

Using Mason's rule, the ratio $\dfrac{a_2}{a_1}$ can be expressed:

$$\frac{a_2}{a_1} = \frac{s_{21}}{1 - \left(T_u\, s_{11} + s_{22}\, T_s + s_{21}\, s_{12}\, T_u\, T_s\right) + \left(T_u\, s_{11}\right)\left(T_s\, s_{22}\right)}$$

$$\approx \frac{s_{21}}{1 - \left(T_u\, s_{11} + T_s\, s_{22} + s_{12}\, s_{21}\, T_u\, T_s\right)}$$

This formula determines the attenuation between $a_1$ and $a_2$ . The minimum and maximum attenuation is given by:

$$Att_{\min} = \frac{s_{21}}{1 + \left(|T_u|\,|s_{11}| + |T_s|\,|s_{22}| + |T_u|\,|T_s|\,|s_{21}|\,|s_{12}|\right)}$$

$$Att_{\max} = \frac{s_{21}}{1 - \left(|T_u|\,|s_{11}| + |T_s|\,|s_{22}| + |T_u|\,|T_s|\,|s_{21}|\,|s_{12}|\right)}$$

The above assumed no phase information and constitute the worst case attenuation.

**EXAMPLE 6 : HARMONIC MEASUREMENT**

For a coaxial attenuator,

$$s_{21} = s_{12}$$
$$= 10^{\left(\frac{-Att\,(dB)}{20}\right)}$$

This modifies the minimum and maximum attenuation equations to become:

$$Att_{min} = \frac{10^{\left(\frac{-Att\,(dB)}{20}\right)}}{1 + \left[\left|T_u\right|\left|s_{11}\right| + \left|T_s\right|\left|s_{22}\right| + 10^{\left(\frac{-Att\,(dB)}{10}\right)}\left|T_u\right|\left|T_s\right|\right]}$$

$$Att_{max} = \frac{10^{\left(\frac{-Att\,(dB)}{20}\right)}}{1 - \left[\left|T_u\right|\left|s_{11}\right| + \left|T_s\right|\left|s_{22}\right| + 10^{\left(\frac{-Att\,(dB)}{10}\right)}\left|T_u\right|\left|T_s\right|\right]}$$

The maximum and minimum attenuation equations are linear and include uncertainty caused by mismatch. The attenuation $Att_{max}$, $Att_{min}$ and $s_{21}$ are squared to convert the attenuation into power form. The power attenuation uncertainty is given by:

$$u(Att) = \frac{\left(Att_{min/max}\right)^2 - s_{21}^2}{s_{21}^2}$$

$$u(Att) = \left[\frac{\left(Att_{min/max}\right)^2}{s_{21}^2} - 1\right]$$

$$= \left\{\frac{1}{\left[1 \pm \left(\left|T_u\right|\left|s_{11}\right|\right) + \left|T_s\right|\left|s_{22}\right| + 10^{-\left(\frac{Att\,(dB)}{10}\right)}\left|T_s\right|\left|T_u\right|\right]^2} - 1\right\}$$

**EXAMPLE 6 : HARMONIC MEASUREMENT**

Hence, for a unit under test SWR of 2.0 : 1, spectrum analyzer's SWR of 1.9 : 1 and attenuator's SWR ( $s_{11}$ and $s_{22}$ ) of 1.015 : 1, the uncertainty is computed as :

$$|T_u| = \frac{2-1}{2+1}$$
$$= 0.3333$$

$$|T_s| = \frac{1.9-1}{1.9+1}$$
$$= 0.3103$$

$$|s_{11}| = |s_{22}|$$
$$= \frac{1.015-1}{1.015+1}$$
$$= 0.0075$$

$$u(Att) = \left\{ \frac{1}{\left[1 \pm \left(0.3333 \times 0.0075 + 0.3103 \times 0.0075 + 10^{-\left(\frac{10}{10}\right)} \times 0.3103 \times 0.3333\right)\right]^2} - 1 \right\}$$

$$u(Att) = 3.1\%, \quad -2.9\%$$

This corresponds to a 10 x log (1.031) = 0.13 dB uncertainty due to mismatch. To convert his uncertainty into the standard derivation form, this value has to be divided by a factor of √2. This gives a standard uncertainty of :

$$u(Att_A) = \frac{0.031}{\sqrt{2}}$$
$$= 0.02192 \frac{Watt}{Watt}$$

**EXAMPLE 6 : HARMONIC MEASUREMENT**

6.2     $u(E_{flatness})$ Derivation

The uncertainty is contributed by the cable and attenuation flatness. The worst case will be the sum of the two terms. To obtain the standard uncertainty, a normalise factor of $\frac{1}{\sqrt{3}}$ is used. The uncertainty is:

$$u(E_{flatness}) = \frac{10^{\left(\frac{1.2\,dB + 0.5\,dB}{10}\right)} - 1}{\sqrt{3}}$$

$$= 0.2766$$

6.3     $u(SA_{error})$ Derivation

The spectrum analyzer uncertainty is given as 1.72dB. The standard uncertainty will be 1.72dB multiple by a normalise factor of $\frac{1}{\sqrt{3}}$ .

$$u(SA_{error}) = \frac{10^{\left(\frac{1.72\,dB}{10}\right)} - 1}{\sqrt{3}}$$

$$= 0.2805$$

6.4     Overall Type B uncertainty

The measured harmonic is 40dBc. In linear form , this is equal to

$$\frac{Offset\_power}{Carrier\_power} = 10^{\left(\frac{-40\,dB}{10}\right)}$$

$$= 1 \times 10^{-4} \frac{Watt}{Watt}$$

$$u(Har) = \frac{Offset\_power}{Carrier\_power} \times \sqrt{\left[\left(\frac{u(Att_A)}{Att_A}\right)^2 + \left(\frac{u(Att_B)}{Att_A}\right)^2 + u^2(Cable_{flatness}) + u^2(SA_{error})\right]}$$

$$= 1 \times 10^{-4} \times \sqrt{\left(\frac{0.02192}{10}\right)^2 + \left(\frac{0.02192}{10}\right)^2 + (0.2766)^2 + (0.2805)^2}$$

$$= 3.9395 \times 10^{-5}$$

Convert to dB, we have the uncertainty as:

$$10 \times \log(1 \times 10^{-4} + 3.9395 \times 10^{-5}) + 40\,dB = 1.442\,dB$$

**EXAMPLE 6 : HARMONIC MEASUREMENT**

## 7.0 UNCERTAINTY BUDGET TABLE

| Sources of Uncertainty | Symbol $x_i$ | Type | Uncertainty Value | Probability Distribution | Coverage Factor | Standard Uncertainty $u(x_i)$ | Sensitivity Coefficient $c_i$ | $|c_i| \times u(x_i)$ | Degrees of freedom $\nu$ |
|---|---|---|---|---|---|---|---|---|---|
| Attenuator A (include mismatch) | $u(Att_A)$ | B | 0.031 | U-shape | $\sqrt{2}$ | 0.02192 | $10^{-5}$ | $0.02\times10^{-5}$ | $\infty$ |
| Attenuator B (include mismatch) | $u(Att_B)$ | B | 0.031 | U-shape | $\sqrt{2}$ | 0.02192 | $10^{-5}$ | $0.02\times10^{-5}$ | $\infty$ |
| Spectrum Analyzer | $u(SA_{Error})$ | B | 0.4859 | Rectangular | $\sqrt{3}$ | 0.2805 | $10^{-4}$ | $2.8\times10^{-5}$ | $\infty$ |
| Path Variation | $u(E_{flatness})$ | B | 0.4791 | Rectangular | $\sqrt{3}$ | 0.2766 | $10^{-4}$ | $2.76\times10^{-5}$ | $\infty$ |
| Repeatability | $u_A(Har)$ | A | - | - | - | 0.0316 dB | 1 | | 4 |

## 8.0 COMBINED STANDARD UNCERTAINTY

The combined uncertainty,

$$u_c(Har) = \sqrt{u_A^2(Har) + u_B^2(Har)}$$
$$= \sqrt{0.0316^2 + 1.442^2}$$
$$= 1.442 \quad dB$$

## 9.0 EFFECTIVE DEGREES OF FREEDOM $\nu_{eff}$

The effective degrees of freedom will be

$$\nu_{eff} = \frac{u_c^4(Har)}{\dfrac{u_A^4(Har)}{\nu_A} + \dfrac{u_B^4(Har)}{\nu_B}}$$
$$= \frac{1.442^4}{\dfrac{0.0316^4}{4} + \dfrac{1.442^4}{\infty}}$$
$$\approx \infty$$

**EXAMPLE 6 : HARMONIC MEASUREMENT**

## 10.0    EXPANDED UNCERTAINTY

For an effective degrees of freedom $v_{eff} = \infty$, $k = 2$ is used at approximately 95 % level of confidence. Hence, the expanded uncertainty

$$U = k\,u_c(Har)$$
$$= 2\,x\,1.442$$
$$= 2.884 \;\; dB$$

## 11.0    REPORTING OF RESULT

The measured harmonic is 40 dBc.  The expanded uncertainty of the measurement is $\pm$ 2.884 dB at a confidence level of approximately 95 % with a coverage factor of $k = 2$.

## EXAMPLE 7: POWER REFERENCE OUTPUT TEST

### 1.0 INTRODUCTION

The power reference oscillator output is factory adjusted to 1 mW ± 0.7 %. To achieve this accuracy, a special measurement system accurate to 0.5 % and allows for a transfer error of ± 0.2 % in making the adjustment is used. The power reference oscillator can be adjusted to ± 0.7 % with the same equipment setup in the calibration laboratory. In order to ensure the maximum accuracy in verifying the power reference oscillator output, the manufacturer's performance test procedures must be followed.
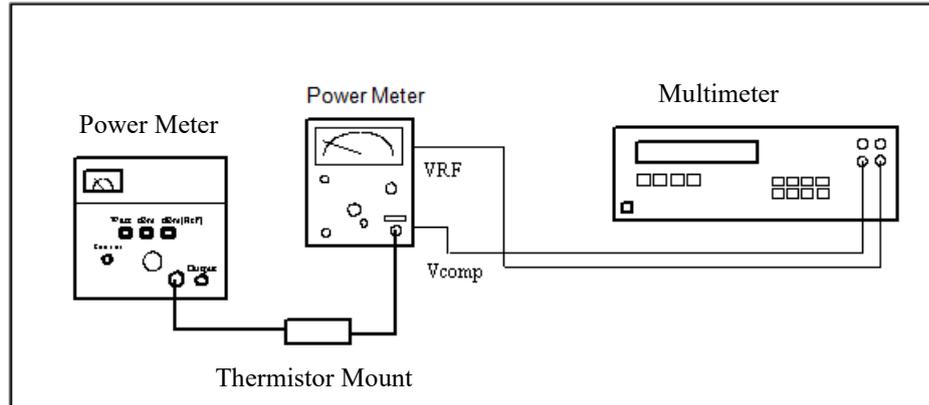
Figure 1: Measurement set-up

1.1 Equipment Used:
  Power Meter
  Thermistor Mount
  Multimeter

1.2 Specifications:
  Internal 50 MHz oscillator factory set to 1 mW ± 0.7 %.
  Accuracy : ± 1.2 % worst case for one year (0 °C to 55 °C).

### 2.0 MEASUREMENT UNCERTAINTY ANALYSIS:

To calculate the power reference oscillator output level (P) :

$$p_{RF} = \frac{2V_{comp}(V_1 - V_0) + V_0^2 - V_1^2}{4R(CALIBRATION\_FACTOR)}$$

## EXAMPLE 7: POWER REFERENCE OUTPUT TEST

where,

$p_{RF}$ : Power reference oscillator output level

$V_{comp}$ : Previously recorded value (Typical value : 4.980529 V)

$V_1$ : Previously recorded value (Typical value : 81.07153 mV)

$V_0$ : Previously recorded value (Typical value : 0.024 mV)

$R$ : Previously recorded value (Typical value : 200.48 $\Omega$)

CALIBRATION FACTOR = Value for thermistor mount at 50 MHz.
(Typical value : 0.9992)

The thermistor mount when connected to the Power reference output of the Power Meter produces $V_{comp}$ and $V_{RF}$ ($V_0$ and $V_1$). A multimeter is used to measure the voltages $V_{comp}$ and $V_{RF}$ ($V_0$ and $V_1$).

2.1    Type A Uncertainty Evaluation, $u_A(P_{RF})$

In this test the power reference is obtained indirectly using the formula above. Therefore, the measurements made from the multimeter is used to evaluate the uncertainty.

The following steps are taken to ensure accurate measurement :

1) The voltage measuring terminals are at the power meter; therefore it is located in an area of free moving air where it is not affected by local heat source.

2) The power meter is then stabilized at its operating temperature.

3) The multimeter is properly set up according to the operating manual to achieve optimum conditions and it is stabilized at its operating temperature.

4) The voltage values are measured, taking five measurements with a complete breakdown between them.

Table of readings

| measurement ($q$) | $R$ | $V_{comp}$ | $V_0$ | $V_1$ | $P_{RF}$ (calculated) |
|---|---|---|---|---|---|
| 1 | 200.27 $\Omega$ | 5.15 V | 71.67 µV | 78.35 mV | $P_{RF1}$ |
| 2 | 200.28 $\Omega$ | 5.15 V | 93.60 µV | 78.37 mV | $P_{RF2}$ |
| 3 | 200.27 $\Omega$ | 5.14 V | 75.67 µV | 78.28 mV | $P_{RF3}$ |
| 4 | 200.27 $\Omega$ | 5.15 V | 83.25 µV | 78.34 mV | $P_{RF4}$ |
| 5 | 200.27 $\Omega$ | 5.16 V | 79.67 µV | 78.35 mV | $P_{RF5}$ |

**EXAMPLE 7: POWER REFERENCE OUTPUT TEST**

$$\text{Mean of } P_{RF} = \overline{q}$$

$$= \frac{1}{5}\sum_{k=1}^{5} q_k$$

$$= \frac{(P_{RF1} + P_{RF2} + P_{RF3} + P_{RF4} + P_{RF5})}{5}$$

$$= \frac{(0.99999 + 0.99998 + 0.99999 + 1.0 + 0.9998)\times 10^{-3}}{5}$$

$$= 0.99998 \times 10^{-3} \text{ Watts}$$

$$\text{Experimental standard deviation} = s(q_k)$$

$$= \sqrt{\frac{1}{4}\sum_{k=1}^{5}\left(q_k - \overline{q}\right)^2}$$

$$= 8.3667 \times 10^{-9} \text{ Watts}$$

$$\text{Experimental SD of the mean} = \frac{s(q_k)}{\sqrt{5}}$$

$$= \frac{(8.3667 \times 10^{-9})}{\sqrt{5}}$$

$$= 3.742 \times 10^{-9} \text{ Watts}$$

$$= u_A(P_{RF})$$

2.2     Type B Uncertainty Evaluation, $u_B(P_{RF})$

2.2.1   Power Meter Accuracy

$\pm$ 1 % of full scale on all ranges (+ 0 °C to 55 °C)

2.2.2   Thermistor Mount

From calibration report:
          12 months calibration cycle
          Frequency : 50 MHz
          Serial No : 84226
          Calibration Factor = 99.92 % $\pm$ 0.34 %
          (The $\pm$ 0.34 % is the uncertainty value dCF of the calibration factor CF)

Effective Calibration Factor uncertainty : 0.9992 x 0.0034 = **0.0034**
dCF = 0.0034

## EXAMPLE 7: POWER REFERENCE OUTPUT TEST

Taking Rectangular distribution:

$$\text{Standard uncertainty, } u_5(CF) = \frac{0.0034}{\sqrt{3}}$$
$$= 0.002$$

From measurement:

Thermistor Mount internal bridge resistance : $200\,\Omega$
The resistance is measured from the multimeter, range of $1\,\text{k}\Omega$.

Standard uncertainty, $u_4(R) = 8.1\ \text{m}\Omega$

This is the measurement uncertainty of the multimeter in the range 1 kW when measuring resistance where we consider a worst case of the measurement being done on the 90th day after calibration date to include a worst case (calibration interval 90 days)

### 2.2.3 Multimeter

Accuracy : ppm of Reading + ppm of Range

| Range | 1 Year |
|---|---|
| 100 mV | 9 + 3 |
| 1 V | 8 + 0.3 |
| 10 V | 8 + 0.05 |
| 100 V | 10 + 0.3 |
| 1000 V | 10 + 0.1 |

Multimeter Specification

| Voltage | Reading | Range | Uncertainty |
|---|---|---|---|
| $V_0$ | 0.024 mV | 100 mV | 0.3 μV |
| $V_1$ | 81.07 mV | 100 mV | 1.03 μV |
| $V_{comp}$ | 4.98 V | 10 V | 0.041 mV |

| Symbol | Uncertainty | Standard Uncertainty |
|---|---|---|
| $u_1(V_0)$ | 0.3 μV | 173.2 nV |
| $u_2(V_1)$ | 1.03 μV | 0.59 μV |
| $u_3(V_{comp})$ | 0.041 mV | 0.024 mV |

Taking 95 % confidence interval with coverage factor $k = 2$.
Assuming rectangular distribution.

**EXAMPLE 7: POWER REFERENCE OUTPUT TEST**

2.2.4   Power Reference Output uncertainty due to mismatch uncertainty of power meter and thermistor mount reflection coefficient

<u>Power Reference Output SWR</u>

SWR : 1.05

Reflection Coefficient : $\dfrac{SWR - 1}{SWR + 1}$

$\qquad\qquad = \underline{0.0244}$   (From specifications)

<u>Thermistor Mount</u>

Worst case reflection coefficient is 0.007 (Sum of mismatch and mismatch error = 0.001 + 0.006)

Mismatch thermistor mount to Power Reference Output in log scale:

$\qquad$ 20 Log( 1 + 0.0244 x 0.007 ) = 0.001483
$\qquad$ 20 Log( 1 - 0.0244 x 0.007 ) = -0.001483

Power Uncertainty due to mismatch  $= 1 \text{ mW} - ( 10^{\frac{0.0014}{10}} \text{ x 1 mW} )$
$\qquad\qquad\qquad\qquad\qquad\qquad = 3.414 \text{ x } 10^{-7} \text{ W}$

$$\textbf{Standard uncertainty} = \frac{3.414 \times 10^{-7} \text{ W}}{\sqrt{2}}$$
$$= 2.414 \text{ nW}$$

Assuming U-Shaped distribution
Assuming Sensitivity Coefficient equal to 1. $c_6 = 1$.

**EXAMPLE 7: POWER REFERENCE OUTPUT TEST**

### 3.0   MEASUREMENT UNCERTAINTY BUDGET

Formula given :

$$P_{RF} = \frac{2V_{comp}(V_1 - V_o) + V_o^2 - V_1^2}{4R(CALIBRATION\_FACTOR)}$$

From the formula, the power reference level is dependent on a few variables. They are $V_0$, $V_1$, $V_{comp}$, $R$ and Calibration Factor. We have to analyze the effect of each variable to the measured power reference level.

The best way to analyze the effect of each variable deviation on power reference level is to perform partial differentiation. Differentiate $P_{RF}$ with respect to one variable at a time, then Root-Sum-Square all uncertainties to calculate the combined standard uncertainty.

$$P_{RF} = \frac{2V_{comp}(V_1 - V_o) + V_o^2 - V_1^2}{4R(CALIBRATION\_FACTOR)}$$

### 4.0   SENSITIVITY COEFFICIENTS

$$
\begin{aligned}
\frac{\partial P_{RF}}{\partial V_0} &= \frac{-V_{comp} + V_0}{2R(CALIBRATION\_FACTOR)} \\
&= \frac{-4.98\ \text{V} + 0.024\ \text{mV}}{2(200)(0.9992)} \\
&= -12.55 \times 10^{-3}\ \text{A}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial P_{RF}}{\partial V_1} &= \frac{V_{comp} - V_1}{2R(CALIBRATION\_FACTOR)} \\
&= \frac{4.98\ \text{V} - 81.07\ \text{mV}}{2(200)(0.9992)} \\
&= 12.26 \times 10^{-3}\ \text{A}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial P_{RF}}{\partial V_{comp}} &= \frac{V_1 - V_0}{2R(CALIBRATION\_FACTOR)} \\
&= \frac{81.07\ \text{mV} - 0.024\ \text{mV}}{2(200)(0.9992)} \\
&= 202.78 \times 10^{-3}\ \text{A}
\end{aligned}
$$

**EXAMPLE 7: POWER REFERENCE OUTPUT TEST**

$$\frac{\partial P_{RF}}{\partial R} = \frac{-\left[2V_{comp}(V_1 - V_0)\right] + V_0^2 - V_1^2}{4R^2(CALIBRATION\_FACTOR)}$$

$$= \frac{-\left[2(4.98)(81.07 \text{ x } 10^{-3} \text{ V} - 0.024 \text{ x } 10^{-3} \text{ V})\right] + (0.024 \text{ x } 10^{-3} \text{ V})^2 - (81.07 \text{ x } 10^{-3} \text{ V})^2}{4\,(200)^2\,(0.9992)}$$

$$= -5.01 \text{ x } 10^{-6} \text{ A}$$

$$\frac{\partial P_{RF}}{\partial CF} = \frac{-\left[2V_{comp}(V_1 - V_0)\right] + V_0^2 - V_1^2}{4R^2(CALIBRATION\_FACTOR)}$$

$$= \frac{-\left[2(4.98)(81.07 \text{ x } 10^{-3} \text{ V} - 0.024 \text{ x } 10^{-3} \text{ V})\right] + (0.024 \text{ x } 10^{-3} \text{ V})^2 - (81.07 \text{ x } 10^{-3} \text{ V})^2}{4\,(200)\,(0.9992)^2}$$

$$= -1.00 \text{ x } 10^{-3} \text{ W} \qquad\qquad (Power = VI)$$

(*CF* abbreviation for *CALIBRATION_FACTOR*)

Let $c_1 = \frac{\partial P_{RF}}{\partial V_o}$, $c_2 = \frac{\partial P_{RF}}{\partial V_1}$, $c_3 = \frac{\partial P_{RF}}{\partial V_{comp}}$, $c_4 = \frac{\partial P_{RF}}{\partial R}$, $c_5 = \frac{\partial P_{RF}}{\partial CF}$, $c_6 = 1$

Where $c_1$, $c_2$, $c_3$, $c_4$, $c_5$ and $c_6$ are Sensitivity Coefficients

## 5.0 UNCERTAINTY BUDGET TABLE

| Source of Uncertainty | Symbol $X_i$ | Type | Uncertainty value | Probability distribution | Coverage factor | Standard uncertainty $u(x_i)$ | Sensitivity coefficient $c_i$ | $\lvert c_i \rvert$ x $u(x_i)$ (W) | Degrees of freedom $v$ |
|---|---|---|---|---|---|---|---|---|---|
| $V_o$ of power meter | $u_1(V)$ | B | 0.3 µV | Rectangular | $\sqrt{3}$ | 173.2x10$^{-9}$ | -12.55x10$^{-3}$ | 2 x 10$^{-9}$ | $\infty$ |
| $V_1$ of power meter | $u_2(V)$ | B | 1.03 µV | Rectangular | $\sqrt{3}$ | 0.59x10$^{-6}$ | 12.26x10$^{-3}$ | 7 x 10$^{-9}$ | $\infty$ |
| $V_{comp}$ of power meter | $u_3(V)$ | B | 0.041 mV | Rectangular | $\sqrt{3}$ | 0.024x10$^{-3}$ | 202.78x10$^{-6}$ | 4 x 10$^{-9}$ | $\infty$ |
| Thermistor Mount Internal bridge Resistance | $u_4(R)$ | B | - | Rectangular | $\sqrt{3}$ | 8.1 x 10$^{-3}$ | -5.01x10$^{-6}$ | 40 x 10$^{-9}$ | $\infty$ |
| Thermistor Mount Calibration Factor | $u_5(CF)$ | B | 0.0034 | Rectangular | $\sqrt{3}$ | 0.002 | -1.00x10$^{-3}$ | 0.000002 | $\infty$ |
| Power Mismatch | $u_6(W)$ | B | 3.41x10$^{-7}$ Watts | U -shaped | $\sqrt{2}$ | 2.41x 10$^{-9}$ | 1 | 2.41x 10$^{-9}$ | $\infty$ |

**EXAMPLE 7: POWER REFERENCE OUTPUT TEST**

## 6.0 COMBINED STANDARD UNCERTAINTY

$$u_c^2(P_{RF}) = u_A^2(P_{RF}) + u_B^2(P_{RF})$$

Type A uncertainty is negligible and hence ignored.

$$u_B^2(P_{RF}) = \sum_{i=1}^{6}(c_i)^2 \, u_i(x_i)$$

$u_1(x_1) = u_1(V) =$ Standard Uncertainty of voltage $V_0$.
$u_2(x_2) = u_2(V) =$ Standard Uncertainty of voltage $V_1$.
$u_3(x_3) = u_3(V) =$ Standard Uncertainty of voltage $V_{comp}$.
$u_4(x_4) = u_4(R) =$ Standard Uncertainty of Thermistor Mount internal bridge resistance.
$u_5(x_5) = u_5(CF) =$ Standard Uncertainty of Thermistor Mount Calibration Factor.
$u_6(x_6) = u_6(W) =$ Standard Power Uncertainty due to mismatch.

$$
\begin{aligned}
u_B^2(P_{RF}) &= (c_1)^2 \, u_1^2(V) + (c_2)^2 \, u_2^2(V) + (c_3)^2 \, u_3^2(V) + (c_4)^2 \, u_4^2(R) + (c_5)^2 \, u_5^2(CF) + (c_6)^2 \, u_6^2(W) \\
&= (-12.55 \times 10^{-3})^2 \, (173.2 \times 10^{-9})^2 + (12.26 \times 10^{-3})^2 \, (0.59 \times 10^{-6})^2 \\
&\quad + (202.78 \times 10^{-6})^2 \, (0.024 \times 10^{-3})^2 + (-5.01 \times 10^{-6})^2 \, (8.1 \times 10^{-3})^2 \\
&\quad + (-1.00 \times 10^{-3})^2 \, (0.002)^2 + (1)^2 \, (2.414 \times 10^{-9})^2 \\
&= 4.003 \times 10^{-12} \\
u_B(P_{RF}) &= 2.0 \times 10^{-6}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u_c(P_{RF}) &= \sqrt{u_A^2(P_{RF}) + u_B^2(P_{RF})} \\
&= 2.0 \ \ \mu W
\end{aligned}
$$

## 7.0 EFFECTIVE DEGREES OF FREEDOM $\nu_{eff}$

The effective degrees of freedom,

$$
\begin{aligned}
\nu_{eff} &= \frac{u_c^4(P_{RF})}{\dfrac{u_A^4(P_{RF})}{\nu_A} + \dfrac{u_B^4(P_{RF})}{\nu_B}} \\
&= \frac{(2.0 \times 10^{-6})^4}{\dfrac{(3.742 \times 10^{-9})^4}{4} + \dfrac{(2.0 \times 10^{-6})^4}{\infty}} \\
&= \infty
\end{aligned}
$$

### EXAMPLE 7: POWER REFERENCE OUTPUT TEST

**8.0      EXPANDED UNCERTAINTY**

For an effective degrees of freedom $v_{eff} = \infty$, $k = 2$ is used at approximately 95 % level of confidence. Therefore, the expanded uncertainty

$$U = k\, u_c\, (P_{RF})$$
$$= 2 \times 2.0 \ \mu W$$
$$= 4.0 \ \mu W$$

**9.0      REPORTING OF RESULT**

The power reference of the power meter is specified as 1 mW. The expanded uncertainty of measurement is $\pm$ 4 $\mu$W at a confidence level of approximately 95 % with a coverage factor of $k = 2$.

**EXAMPLE 8: TRANSITION TIME TEST**

## 1.    INTRODUCTION

The transition time is normally measured at 10% to 90% of amplitude. To achieve this measurement on a digital oscilloscope, a programmable 500 MHz Pulse Generator whereby all output signals have a fixed transition time ≤ 200 ps is used. The Pulse Generator then applies a fast rise time pulse to the Digitizing Oscilloscope and makes an automatic rise-time measurement.
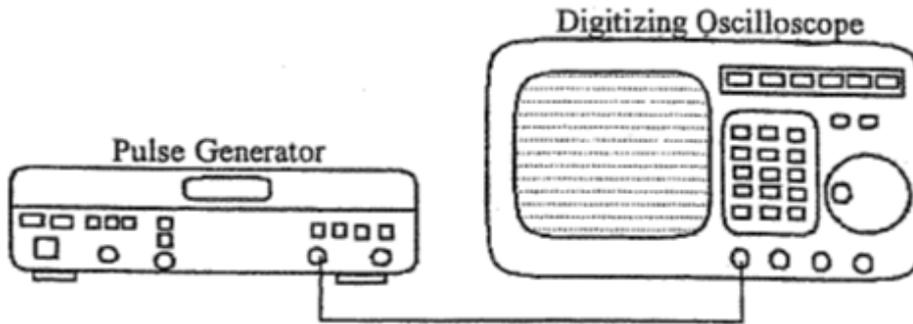


Figure 1: Measurement set-up

1.1    Equipment Used:
       500 MHz Pulse Generator

1.2    Specifications:
       The Digitizing Oscilloscope has a repetitive transition time that is ≤ 700 ps

## 2.0    MEASUREMENT UNCERTAINTY ANALYSIS

To calculate the Rise-Time of the signal observed in the scope:

$$RT_{Observed} = \sqrt{RT_{Scope}^2 + RT_{Gen}^2}$$

where,
$RT_{Observed}$  : Rise-time of signal observed on oscilloscope
$RT_{Scope}$     : Rise-time of the oscilloscope itself
$RT_{Gen}$       : Rise-time of the Pulse of the Pulse Generator

**EXAMPLE 8: TRANSITION TIME TEST**

Since we are to determine the Rise-time of the Digitizing Oscilloscope, we shall rearrange the equation. Therefore, the Measurement Equation we are using for Rise-time of the Scope is :

$$RT_{Scope} = \sqrt{RT_{Observed}^2 - RT_{Gen}^2}$$

There is a reminder to take note of is that the rise-time of the oscilloscope is not to be more than 5 times as fast as the signal rise-time as the displayed rise-time will become slower than actual.

2.1     Type A Uncertainty Evaluation  $u_A$ ($RT_{Scope}$)

In this test the Rise-time of the oscilloscope is obtained indirectly using the formula above; therefore the measurements made from the oscilloscope are used to evaluate the uncertainty.

The following steps are taken to ensure accurate measurement :

1.      The Oscilloscope and Pulse Generator are to be located in an area of free moving air where they are not affected by local heat source.

2.      The pulse generator is stabilized at its operating temperature of 0 °C to +55 °C with a 30 minutes warm up phase with a 50 ohms load resistance at all outputs.

3.      The oscilloscope is properly set up according to the operating manual to achieve optimum conditions and it is stabilized at its operating temperature +15°C to +55 °C with a 30 minutes Power On warm up.

4.      The rise-time values are measured and taking five measurements with a complete breakdown between them.

Table of readings

| Measurement (i) | $RT_{Observed}$ | $RT_{Gen}$ | $RT_{Scope}$ (Calculated) | $RT_{si}$ |
|---|---|---|---|---|
| 1 | 540 ps | 200 ps | 501.60 ps | $RT_{s1}$ |
| 2 | 539 ps | 200 ps | 500.52 ps | $RT_{s2}$ |
| 3 | 540 ps | 200 ps | 501.60 ps | $RT_{s3}$ |
| 4 | 540 ps | 200 ps | 501.60 ps | $RT_{s4}$ |
| 5 | 539 ps | 200 ps | 500.52 ps | $RT_{s5}$ |

**EXAMPLE 8: TRANSITION TIME TEST**

$$Mean\,of\,RT_{Scope} = \bar{q}$$
$$= \frac{1}{5}\sum_{k=1}^{5}q_k$$
$$= \frac{(RT_{s1} + RT_{s2} + RT_{s3} + RT_{s4} + RT_{s5})}{5}$$
$$= \frac{(501.60 + 500.52 + 501.60 + 501.60 + 500.52)}{5}$$
$$= \frac{2505.84}{5}$$
$$= 501.17\,ps$$

Experimental standard deviation,

$$s(q_k) = \sqrt{\frac{1}{4}\left[\sum_{k=1}^{5}\left(q_k - \bar{q}\right)^2\right]}$$
$$= 0.59\,ps$$

Experimental SD of the mean,

$$s\left(\bar{q}_k\right) = \frac{0.59\,ps}{\sqrt{5}}$$
$$= 0.26\,ps$$
$$= u_A\left(RT_{Scope}\right)$$

2.2     Type B Uncertainty Evaluation $u_B$ ($RT_{Scope}$)

2.2.1   Pulse Generator Transition Time Specification

10% to 90% amplitude : $\leq 200$ ps, 300 mV to 3V Range

Standard Uncertainty of Pulse Generator

$$u_{Gen} = \frac{200\,ps}{\sqrt{3}}$$
$$= 115.47\,ps$$

(The probability distribution of the rise-time specification is not exactly known. Hence, rectangular distribution is adopted as given in GUM)

2.2.2   The Rise-time of the observed Signal

The reading observed is 539 ps and the uncertainty present here is only the resolution of the displayed reading of the Rise-time value and it is considered negligible.

**EXAMPLE 8: TRANSITION TIME TEST**

### 3.0 MEASUREMENT UNCERTAINTY BUDGET

Formula given :

$$RT_{Scope} = \sqrt{RT_{Observed}^2 - RT_{Gen}^2}$$

From the formula, the Rise-time of scope is now dependent only on the standard uncertainty of the Pulse Generator.

The best way to analyze the effect of the variable deviation on the Rise-time is to perform partial differentiation. Differentiate $RT_{Scope}$ with respect to $RT_{Gen}$, then Root-Sum-Square the uncertainty to calculate the combined standard uncertainty.

Therefore,

$$RT_{Scope} = \sqrt{RT_{Observed}^2 - RT_{Gen}^2}$$

### 4.0 SENSITIVITY COEFFICIENTS

Taking partial derivatives of the equation, the sensitivity coefficients

$$c_1 = \frac{\partial RT_{Scope}}{\partial RT_{Gen}}$$

$$= \frac{\partial \sqrt{RT_{Observed}^2 - RT_{Gen}^2}}{\partial RT_{Gen}}$$

$$= \frac{-RT_{Gen}}{\sqrt{RT_{Observed}^2 - RT_{Gen}^2}}$$

Substituting $RT_{Observed} = 539$ ps and $RT_{Gen} = 200$ ps

$$c_1 = \frac{-200}{\sqrt{539^2 - 200^2}}$$

$$= -0.40$$

**EXAMPLE 8: TRANSITION TIME TEST**

**5.0     UNCERTAINTY BUDGET TABLE**

| Source of Uncertainty | Symbol $x_i$ | Type | Uncertainty Value | Probability distribution | Coverage factor | Standard Uncertainty $u(x_i)$ | Sensitivity coefficient $c_i$ | $|c_i|xu(x_i)$ | Degrees of freedom $v$ |
|---|---|---|---|---|---|---|---|---|---|
| Rise-time of pulse generator | $U_{Gen}$ (t) | B | 200 ps | Rectangular | $\sqrt{3}$ | 115.47 ps | -0.40 ps | 46.19 | ∞ |
| Rise time of oscilloscope | $u_B$ ($RT_{Scope}$) | B | - | t-distribution | - | 46.19 ps | | | ∞ |

**6.0     COMBINED STANDARD UNCERTAINTY**

$$u_c^2\left(RT_{Scope}\right) = u_A^2\left(RT_{Scope}\right) + u_B^2\left(RT_{Scope}\right)$$

The Type A uncertainty is negligible and hence ignored

$$
\begin{aligned}
u_B\left(RT_{Scope}\right) &= \sqrt{c_1^2 u_{Gen}^2} \\
&= \sqrt{-0.40 x 115.47}\, ps \\
&= 46.19\, ps
\end{aligned}
$$

Hence,

$$
\begin{aligned}
u_c\left(RT_{Scope}\right) &= \sqrt{u_A^2\left(RT_{Scope}\right) + u_B^2\left(RT_{Scope}\right)} \\
&= 46.19\, ps
\end{aligned}
$$

**7.0     EFFECTIVE DEGREES OF FREEDOM $v_{eff}$**

The effective degrees of freedom

$$
\begin{aligned}
\upsilon_{eff} &= \frac{u_c^4\left(RT_{Scope}\right)}{\dfrac{u_A^4\left(RT_{Scope}\right)}{\upsilon_A} + \dfrac{u_B^4\left(RT_{Scope}\right)}{\upsilon_B}} \\
\\
&= \frac{\left(46.19 x 10^{-12}\right)^4}{\dfrac{\left(0.26 x 10^{-12}\right)^4}{4} + \dfrac{\left(46.19 x 10^{-12}\right)^4}{\infty}} \\
\\
&= \infty
\end{aligned}
$$

**EXAMPLE 8: TRANSITION TIME TEST**

## 8.0    EXPANDED UNCERTAINTY

For an effective degrees of freedom, $v_{eff} = \infty$, $k = 2$ is used at approximately 95% level of confidence. Therefore, the expanded uncertainty

$$U = ku_c\left(RT_{Scope}\right)$$
$$= 2 x 46.19$$
$$= 92.38\, ps$$

## 9.0    REPORTING OF RESULT

The rise-time of the digital oscilloscope is specified as $\leq$ 700 ps. The uncertainty of measurement is $\pm$ 92.38 ps at a confidence level of approximately 95% with a coverage factor $k = 2$

**EXAMPLE 9: CALIBRATION OF AN INDUSTRIAL PRESSURE GUAGE**

**1.0     INTRODUCTION**

The increasing and decreasing gauge pressure indications of the Industrial Pressure Gauge are calibrated by comparing against a Pressure Test Gauge.
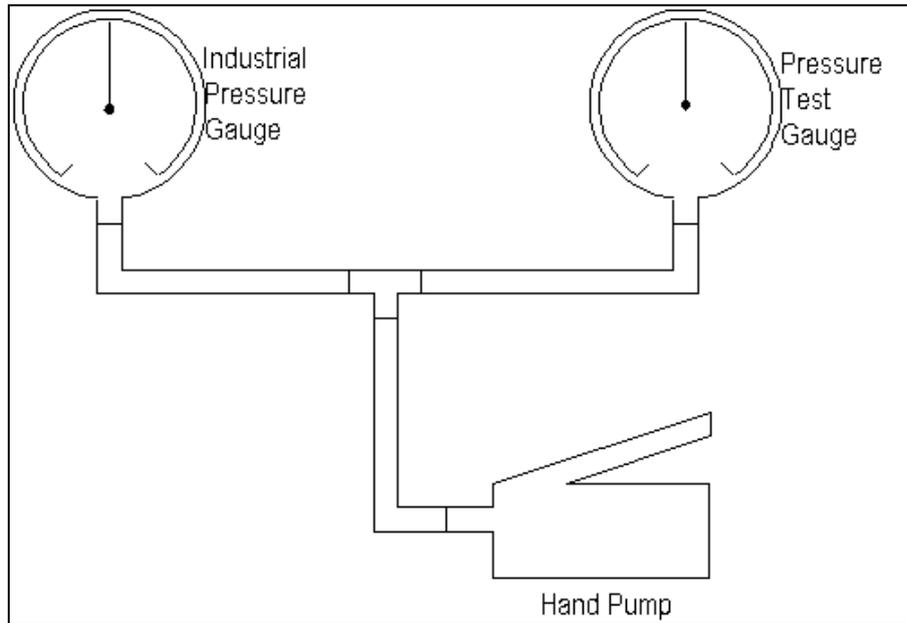


Figure 1: Set-up of an industrial pressure gauge calibration

The unit-under-test (UUT) is an Industrial Pressure Gauge with:

    Range : 0 - 100 psi
    Scale: 1 div  = 1 psi
    Resolution: ½ div = 0.5 psi
    Accuracy class: 1

It is calibrated against a Pressure Test Gauge with:

    Range : 0 - 100 psi
    Scale: 1 div  = 0.2 psi
    Resolution: ½ div = 0.1 psi
    Uncertainty: ± 0.25% span

# EXAMPLE 9: CALIBRATION OF AN INDUSTRIAL PRESSURE GUAGE

## 2.0    UUT MEASURED DATA

The number of readings for each test point, n = 5, are tabulated as below:

### Table 1 - Increasing Gauge Pressure

| Nominal Pressure as from Test Gauge | Readings of UUT (psi) | | | | | Mean | Standard Deviation |
|---|---|---|---|---|---|---|---|
| (psi) | 1 | 2 | 3 | 4 | 5 | (psi) | (psi) |
| 20 | 20.0 | 20.5 | 20.0 | 20.0 | 20.5 | 20.2 | 0.2739 |
| 40 | 40.0 | 40.5 | 40.5 | 40.5 | 40.0 | 40.3 | 0.2739 |
| 60 | 60.5 | 60.5 | 60.5 | 60.5 | 60.0 | 60.4 | 0.2236 |
| 80 | 80.0 | 80.5 | 80.5 | 80.5 | 80.5 | 80.4 | 0.2236 |
| 100 | 100.0 | 100.0 | 100.5 | 100.5 | 100.5 | 100.3 | 0.2739 |

### Table 2 - Decreasing Gauge Pressure

| Nominal Pressure as from Test Gauge | Readings of UUT (psi) | | | | | Mean | Standard Deviation |
|---|---|---|---|---|---|---|---|
| (psi) | 1 | 2 | 3 | 4 | 5 | (psi) | (psi) |
| 80 | 80.0 | 80.5 | 80.0 | 80.0 | 80.0 | 80.1 | 0.2236 |
| 60 | 59.5 | 59.5 | 59.5 | 59.5 | 60.0 | 59.6 | 0.2236 |
| 40 | 40.5 | 40.0 | 40.0 | 40.5 | 40.5 | 40.3 | 0.2739 |
| 20 | 20.0 | 20.0 | 20.0 | 20.5 | 20.0 | 20.2 | 0.2739 |

## 3.0    MATHEMATICAL MODEL

Hence the mathematical relationship can be modelled as:

$$Y_{UUT} = X_{Test\ Gauge} + E_{Rep} + E_{Res} \qquad ...(1)$$

where

$Y_{UUT}$          : Industrial Pressure Gauge indication.

$X_{Test\ Gauge}$          : Pressure Test Gauge indication.

$E_{Rep}$          : Error due to random variations of the Industrial Pressure Gauge indication when repeatedly applied of same Pressure Test Gauge indication.

$E_{Res}$          : Error due to resolution of the Industrial Pressure Gauge indication.

## EXAMPLE 9: CALIBRATION OF AN INDUSTRIAL PRESSURE GUAGE

### 4.0    UNCERTAINTY EQUATION

Considering all the input quantities are uncorrelated, the combined standard uncertainty equation is then given by:

$$u_c(Y_{UUT}) = \sqrt{\left[\left(\frac{\partial Y_{UUT}}{\partial X_{Test\,Gauge}}\right)u(X_{Test\,Gauge})\right]^2 + \left[\left(\frac{\partial Y_{UUT}}{\partial E_{Rep}}\right)u(E_{Rep})\right]^2 + \left[\left(\frac{\partial Y_{UUT}}{\partial E_{Res}}\right)u(E_{Res})\right]^2} \quad ...(2)$$

where

$u(X_{Test\,Gauge})$    : standard uncertainty for Pressure Test Gauge indication.

$u(E_{Rep})$          : standard uncertainty for random variations of Industrial Pressure Gauge indication.

$u(E_{Res})$          : standard uncertainty for resolution of Industrial Pressure Gauge indication.

$\dfrac{\partial Y_{UUT}}{\partial X_{Test\,Gauge}}$,   $\dfrac{\partial Y_{UUT}}{\partial E_{Rep}}$   and   $\dfrac{\partial Y_{UUT}}{\partial E_{Res}}$   are the partial derivatives of $Y_{UUT}$ with respect to $X_{Test\,Gauge}$, $E_{Rep}$, $E_{Res}$ in equation (1). These are the sensitivity coefficients and all are equal to 1.

### 5.0    TYPE A STANDARD UNCERTAINTY

Maximum standard deviation from Tables 1 and 2 is,

$$s(q_k) = 0.2739 \text{ psi} \quad\quad\quad ...(3)$$

Hence the Type A standard uncertainty is given by

$$u(E_{Rep}) = \frac{s(q_k)}{\sqrt{n}} \quad\quad\quad ...(4)$$
$$= \frac{0.2739}{\sqrt{5}}$$
$$= 0.122 \text{ psi}$$

**EXAMPLE 9: CALIBRATION OF AN INDUSTRIAL PRESSURE GUAGE**

## 6.0 TYPE B STANDARD UNCERTAINTY

6.1 The expanded uncertainty taken from the calibration report of the Pressure Test Gauge is $\pm\,0.25\,\%$ of span with the coverage factor of $k = 2$ at approximately 95 % level of confidence assuming normal distribution.

$$a = \pm\,(0.25\%\,\text{x}\,100)\ \text{psi} \qquad\qquad ...(5)$$
$$= \pm\,0.25\ \text{psi}$$

Hence the Type B standard uncertainty is given by

$$u(X_{Test\,Gauge}) = \frac{a}{2} \qquad\qquad ...(6)$$
$$= \frac{0.25}{2}$$
$$= 0.125\ \text{psi}$$

6.2 The resolution of the Industrial Pressure Gauge indication is half of its divisional interval, which is 0.5 psi and assuming rectangular distribution,

$$u(E_{Res}) = \frac{0.5}{\sqrt{12}} \qquad\qquad ...(7)$$
$$= 0.144\ \text{psi}$$

## 7.0 UNCERTAINTY BUDGET TABLE

| Source of Uncertainty | Symbol $x_i$ | Type | Uncertainty value | Probability distribution | Coverage factor | Standard uncertainty $u(x_i)$ | Sensitivity coefficient $c_i$ | $|c_I|$ x $u(x_i)$ (psi) | Degrees of freedom $v$ |
|---|---|---|---|---|---|---|---|---|---|
| Pressure Test Gauge | $X_{Test\,Gauge}$ | B | 0.25 psi | Normal | 2 | 0.125 psi | 1 | 0.125 | $\infty$ |
| Resolution of Industrial Pressure Gauge | $E_{Res}$ | B | 0.5 psi | Rectangular | $\sqrt{12}$ | 0.144 psi | 1 | 0.144 | $\infty$ |
| Repeatability of Industrial Pressure Gauge | $E_{Rep}$ | A | 0.122 psi | - | - | 0.122 psi | 1 | 0.122 | 4 |

**EXAMPLE 9: CALIBRATION OF AN INDUSTRIAL PRESSURE GUAGE**

### 8.0 COMBINED STANDARD UNCERTAINTY

The combined standard uncertainty is then given by:

$$u_c(Y_{UUT}) = \sqrt{u^2(X_{Test\,Gauge}) + u^2(E_{Res}) + u^2(E_{Rep})} \qquad ...(8)$$
$$= 0.226 \; psi$$

### 9.0 EFFECTIVE DEGREES OF FREEDOM OF COMBINED STANDARD UNCERTAINTY

The effective degrees of freedom of the combined standard uncertainty is given by:

$$v_{eff} = \frac{u_c^4(Y_{UUT})}{\dfrac{\left[1 \times u(X_{Test\,Gauge})\right]^4}{\infty} + \dfrac{\left[1 \times u(E_{Res})\right]^4}{\infty} + \dfrac{\left[1 \times u(E_{Rep})\right]^4}{4}} \qquad ...(9)$$

$$= \frac{0.226^4}{\dfrac{0.125^4}{\infty} + \dfrac{0.144^4}{\infty} + \dfrac{0.122^4}{4}}$$

$$\approx 47$$

### 10.0 EXPANDED UNCERTAINTY

Since the effective degrees of freedom is greater than 30, $k = 2$ at a confidence level of approximately 95 % is used.

Therefore, the expanded uncertainty is given by:

$$U = k\,u_c(Y_{UUT})$$
$$= 2 \times 0.226$$
$$= 0.5 \; psi$$

### 11.0 REPORTING OF RESULTS

Generally, for the Industrial Pressure Gauge indication range of 0 to 100 psi, having an expanded uncertainty of $\pm$ 0.5 psi with a coverage factor of $k = 2$ assuming a normal distribution at a level of confidence of approximately 95 %.

## EXAMPLE 10: TORQUE TESTER CALIBRATION

### 1.0    INTRODUCTION

This example demonstrates how the uncertainty of a calibration point for torque tester is evaluated and computed.
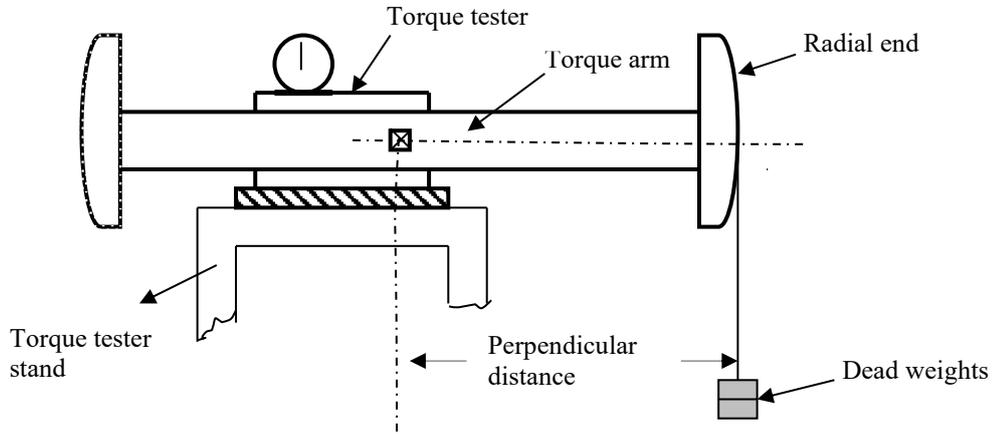


Figure 1 : Set-up of torque tester calibration

The torque tester is calibrated using a torque arm and dead weights as shown in Figure 1. The dead weights are placed at one end of the torque arm, which then produced a desired known torque. The torque arm is designed with a radial end which ensures that the perpendicular distance of the torque arm is maintained, even when it rotates upon loaded.

### 2.0    MATHEMATICAL MODEL

*Measured Torque = Applied Torque + Error*
$$T_m = T_a + e \qquad \qquad \text{...(1)}$$

*Applied Torque*, $T_a$ = *Force* x Perpendicular distance
$$= \text{Mass x Local Gravity  x Perpendicular distance}$$
$$= M\left(1 - \frac{\rho_a}{\rho_m}\right) g \, d_{20} (1 + \alpha \, \theta) \qquad \qquad \text{...(2)}$$

Hence,

$$T_m = M\left(1 - \frac{\rho_a}{\rho_m}\right) g \, d_{20} (1 + \alpha \theta) + e \qquad \qquad \text{...(3)}$$

## EXAMPLE 10: TORQUE TESTER CALIBRATION

where

$M$      : weight of the dead weights applied at the end of the torque arm in kg.

$\rho_a$      : density of air in kg/m$^3$.

$\rho_m$      : density of the dead weight material in kg/m$^3$.

$d_{20}$      : effective length of the torque arm in m at 20 °C as given in its calibration certificate.

$\alpha$      : coefficient of thermal expansion of the torque arm in °C$^{-1}$.

$\theta$      : deviation of temperature from 20 °C reference temperature of the torque arm in °C.

$g$      : local gravity acceleration in m/s$^2$.

$e$      : consists of resolution ($\delta$) and repeatability (T) of the torque tester being calibrated, all in N.m.

Assuming all the above quantities are uncorrelated, equation (3) can be re-written as,

$$T_m = f(M, \rho_a, \rho_m, \alpha, \theta, d_{20}, g, e) \qquad \qquad ...(4)$$

Hence the contributory variances are as follows,

$$u_c^2(T_m) = c_M^2 \, u^2(M) + c_{\rho_a}^2 \, u^2(\rho_a) + c_{\rho_m}^2 \, u^2(\rho_m) + c_\alpha^2 \, u^2(\alpha)$$
$$+ c_\theta^2 \, u^2(\theta) + c_{d_{20}}^2 \, u^2(d_{20}) + c_g^2 \, u^2(g) + c_e^2 \, u^2(e) \qquad ...(5)$$

with sensitivity coefficients,

$$c_M = \frac{\partial f}{\partial M}$$
$$= \left(1 - \frac{\rho_a}{\rho_m}\right) d_{20} \, (1 + \alpha \, \theta) \, g$$

$$c_{\rho_a} = \frac{\partial f}{\partial \rho_a}$$
$$= -\frac{M}{\rho_m} d_{20} (1 + \alpha \, \theta) \, g$$

$$c_{\rho_m} = \frac{\partial f}{\partial \rho_m}$$
$$= M \frac{\rho_a}{\rho_m^2} d_{20} (1 + \alpha \, \theta) \, g$$

## EXAMPLE 10: TORQUE TESTER CALIBRATION

$$c_{d_{20}} = \frac{\partial f}{\partial d_{20}}$$

$$= M\left(1 - \frac{\rho_a}{\rho_m}\right)(1 + \alpha\,\theta)\,g$$

$$c_\alpha = \frac{\partial f}{\partial \alpha}$$

$$= \theta\,d_{20}\,M\left(1 - \frac{\rho_a}{\rho_m}\right)g$$

$$c_\theta = \frac{\partial f}{\partial \theta}$$

$$= \alpha\,d_{20}\,M\left(1 - \frac{\rho_a}{\rho_m}\right)g$$

$$c_g = \frac{\partial f}{\partial g}$$

$$= M\left(1 - \frac{\rho_a}{\rho_m}\right)d_{20}\,(1 + \alpha\,\theta)$$

$$c_e = \frac{\partial f}{\partial e}$$

$$= 1$$

In this example, a calibration point at 12 Nm is chosen on a torque tester which measuring range is 0 to 12 Nm.

## EXAMPLE 10: TORQUE TESTER CALIBRATION

### 3.0 TYPE B UNCERTAINTY EVALUATION

Calculated weight to be applied is 4.910 kg to produce the required known torque. The uncertainty computed using a coverage factor of $k = 2$ is 0.025 kg.

The standard uncertainty is then

$$u(M) = \frac{0.025 \text{ kg}}{2}$$
$$= 12.5 \times 10^{-3} \text{ kg}$$

The Density of dead weights, $\rho_m$ is given by the manufacturer to be 8000 kg/m$^3$ with a rectangular distribution with bounds $\pm$ 100 kg/m$^3$. The standard uncertainty is then

$$u(\rho_m) = \frac{100 \text{ kg/m}^3}{\sqrt{3}}$$
$$= 57.74 \text{ kg/m}^3$$

The local gravity acceleration, $g$ is calibrated to be $9.78065 \pm 0.00005$ m/s$^2$ using a coverage factor of $k = 2$. The standard uncertainty is then

$$u(g) = \frac{0.00005 \text{ m/s}^2}{2}$$
$$= 2.5 \times 10^{-5} \text{ m/s}^2$$

The length of torque arm, $d_{20}$ is calibrated to be $0.25000 \pm 0.00002$ m using a coverage factor of $k = 2$. The standard uncertainty is then

$$u(d_{20}) = \frac{0.00002 \text{ m}}{2}$$
$$= 10 \times 10^{-6} \text{ m}$$

Coefficient of thermal expansion, $\alpha$ of the torque arm is given by the manufacturer to be 1x10$^{-7}$ m / $^{\circ}$C. Since the value of $\alpha$ is very small, its standard uncertainty contribution to the combined uncertainty is neglected, hence

$$u(\alpha) = 0$$

## EXAMPLE 10: TORQUE TESTER CALIBRATION

The temperature of the environment is controlled and monitored to be 20 °C, hence $\theta = 0$. The uncertainty represented by a rectangular distribution with bounds $\pm 1$ °C. The standard uncertainty is then

$$u(\theta) = \frac{1 \ ^\circ C}{\sqrt{3}}$$
$$= 0.5774 \ ^\circ C$$

Within the laboratory controlled ambient condition, the air density of 1.2 kg/m³ would have an uncertainty represented by a rectangular distribution with bounds $\pm 0.06$ kg/m³. The standard uncertainty is then,

$$u(\rho_a) = \frac{0.06 \ \text{kg/m}^3}{\sqrt{3}}$$
$$= 0.03464 \ \text{kg/m}^3$$

Standard uncertainty of indication of torque tester display is,

$$u(\delta) = \frac{\delta}{2\sqrt{3}}$$
$$= \frac{0.01}{2\sqrt{3}}$$
$$= 2.88 \times 10^{-3} \ \text{Nm}$$

where $\delta$ = resolution of torque tester display

## 4.0 TYPE A UNCERTAINTY EVALUATION

10 independent repeated observations were taken. The arithmetic mean of these readings is computed to be 11.85 Nm with a standard uncertainty of

$$u(T) = s\left(\overline{T}\right)$$
$$= \frac{\text{Standard deviation}}{\sqrt{n}}$$
$$= \frac{0.05 \ \text{Nm}}{\sqrt{10}}$$
$$= 1.58 \times 10^{-2} \ \text{Nm}$$

where n = number of independent repeated observations.

$$u^2(e) = u^2(T) + u^2(\delta) \qquad \qquad ...(6)$$

# EXAMPLE 10: TORQUE TESTER CALIBRATION

## 5.0     SENSITIVITY COEFFICIENTS

The sensitivity coefficients in equation (5) are computed as,

$$\frac{\partial f}{\partial \alpha} = 0 \times 0.25 \times 4.91 \left(1 - \frac{1.2}{8000}\right) 9.78065$$
$$= 0$$

$$\frac{\partial f}{\partial \theta} = 1 \times 10^{-7} \times 0.25 \times 4.91 \left(1 - \frac{1.2}{8000}\right) 9.78065$$
$$= 1.20 \times 10^{-6}$$

$$\frac{\partial f}{\partial M} = \left(1 - \frac{1.2}{8000}\right) 0.25 \left(1 + 1 \times 10^{-7} \times 0\right) 9.78065$$
$$= 2.445$$

$$\frac{\partial f}{\partial \rho_a} = -\frac{4.91}{8000} \left[0.25 \left(1 + 1 \times 10^{-7} \times 0\right)\right] 9.78065$$
$$= -0.002$$

$$\frac{\partial f}{\partial \rho_m} = 4.91 \left(\frac{1.2}{8000^2}\right) \left[0.25 \left(1 + 1 \times 10^{-7} \times 0\right)\right] 9.78065$$
$$= 2.25 \times 10^{-7}$$

$$\frac{\partial f}{\partial g} = 4.91 \left(1 - \frac{1.2}{8000}\right) 0.25 \left(1 + 1 \times 10^{-7} \times 0\right)$$
$$= 1.227$$

$$\frac{\partial f}{\partial d_{20}} = 4.91 \left(1 - \frac{1.2}{8000}\right) \left(1 + 1 \times 10^{-7} \times 0\right) 9.78065$$
$$= 48.016$$

# EXAMPLE 10: TORQUE TESTER CALIBRATION

## 6.0 UNCERTAINTY BUDGET TABLE

| Source of Uncertainty | Symbol $x_i$ | Type | Uncertainty value | Probability distribution | Coverage factor | Standard uncertainty $u(x_i)$ | Sensitivity coefficient $c_i$ | $\|c_i\| \times u(x_i)$ Nm | Degrees of freedom $v$ |
|---|---|---|---|---|---|---|---|---|---|
| Weights | $M$ | B | 25 x 10⁻³ kg | Normal | 2 | 12.5 x 10⁻³ kg | 2.445 | 3.06 x 10⁻² | ∞ |
| Density of weights | $\rho_m$ | B | 100 kg/m³ | Rectangular | $\sqrt{3}$ | 57.74 kg/m³ | 2.25 x 10⁻⁷ | 2.25 x 10⁻⁵ | ∞ |
| Gravity acceleration | $g$ | B | 5 x 10⁻⁵ m/s² | Normal | 2 | 2.5 x 10⁻⁵ m/s² | 1.227 | 3.07 x 10⁻⁵ | ∞ |
| Length of arm | $d_{20}$ | B | 2 x 10⁻⁵ m | Normal | 2 | 10 x 10⁻⁶ m | 48.016 | 4.80 x 10⁻⁴ | ∞ |
| Coefficient of Thermal Expansion | $\alpha$ | B | - | - | - | 0 | 0 | 0 | - |
| Temperature | $\theta$ | B | 1 ˚C | Rectangular | $\sqrt{3}$ | 5.774 x 10⁻¹ ˚C | 1.2 x 10⁻⁶ | 6.92 x 10⁻⁷ | ∞ |
| Air density | $\rho_a$ | B | 6 x 10⁻² kg/m³ | Rectangular | $\sqrt{3}$ | 3.464 x 10⁻² kg/m³ | -2 x 10⁻³ | -6.93 x 10⁻⁵ | ∞ |
| Resolution | $\delta$ | B | 5 x 10⁻³ Nm | Rectangular | $\sqrt{3}$ | 2.88 x 10⁻³ Nm | 1 | 2.88 x 10⁻³ | ∞ |
| Repeatability | $T$ | A | 1.58 x 10⁻² Nm | - | - | 1.58 x 10⁻² Nm | 1 | 1.58 x 10⁻² | 9 |

## 7.0 COMBINED STANDARD UNCERTAINTY

Collection of the individual terms discussed previously and substitute them and equation (6) into equation (5) yields for the variance and the combined standard uncertainty, $u_c(T_m)$.

$$u_c^2(T_m) = 2.445^2 \left(12.5 \text{ x } 10^{-3}\right)^2 + \left(-0.002\right)^2 0.03464^2 + \left(2.25 \text{ x } 10^{-7}\right)^2 \text{ x } 57.74^2$$
$$+ 48.016^2 \left(10 \text{ x } 10^{-6}\right)^2 + 0 \text{ x } 0 + \left(1.2 \text{ x } 10^{-6}\right)^2 0.5774^2 + 1.227^2 \left(2.5 \text{ x } 10^{-5}\right)^2$$
$$+ \left(1.58 \text{ x } 10^{-2}\right)^2 + \left(2.88 \text{ x } 10^{-3}\right)^2$$
$$= 11.92 \text{ x } 10^{-4}$$
$$u_c(T_m) = \sqrt{11.92 \text{ x } 10^{-4}}$$
$$= 0.035 \text{ Nm}$$

# EXAMPLE 10: TORQUE TESTER CALIBRATION

## 8.0 EFFECTIVE DEGREES OF FREEDOM

Suppose that one is required to obtain an expanded uncertainty $U = k\,u_c(T_m)$ that provides a level of confidence of approximately 95 %. The procedure is to first compute the effective degrees of freedom, $\nu_{eff}(T_m)$.

To obtain the effective degrees of freedom, $\nu_i$ the degrees of freedom for each standard uncertainty component is required. For a component obtained from Type A evaluation, $\nu_i$ is obtained from the number of independent repeated observations upon which the corresponding input estimate is based and the number of independent quantities determined from those observations.

For a component obtained from a Type B evaluation, $\nu_i$ is obtained from the judged reliability of the value of that component, which is often the case in practise, $\nu_i \to \infty$, such that $\nu(M)$, $\nu(\rho_m)$, $\nu(\alpha)$, $\nu(\theta)$, $\nu(\rho_a)$, $\nu(\delta)$, $\nu(d_{20})$ and $\nu(g)$ are treated as $\to \infty$. Hence,

$$\nu_{eff}(T_m) = \frac{[u_c(T_m)]^4}{\dfrac{u^4(T)}{\nu(T)}}$$

$$= \frac{0.035^4}{\dfrac{(1.58 \times 10^{-2})^4}{9}}$$

$$= 216$$

## 9.0 EXPANDED UNCERTAINTY

For $\nu_{eff}(T_m) = 216$, $k = 2$ can be used at approximately 95 % level of confidence.

$$Expanded\ Uncertainty, U = k\,u_c(T_m)$$
$$= 2 \times 0.035$$
$$= 0.07\ Nm$$

## 10.0 REPORTING OF RESULTS

The measured torque of 11.85 Nm having an expanded uncertainty of $\pm\,0.07$ Nm with a coverage factor of $k = 2$, assuming a normal distribution at a level of confidence of approximately 95 %.

# EXAMPLE 11: CALIBRATION OF WEIGHING MACHINE

## 1.0 INTRODUCTION

An electronic weighing machine with a scale of range, 0 to 60 kg and resolutions of 0.01 kg was calibrated by direct comparison method, using a set of OIML class $M_2$ standard weight. Prior to calibration, the weighing machine was adjusted at full load using a standard weight according to the manufacturer procedure. The repeatability of the weighing machine was determined at near zero, half-full load, full load and the pool standard deviation was obtained. Normally the scale reading of the weighing machine would be calibrated by placing standard weights on the receptor in increasing increments of about 10% of the full capacity. In this example the measurement uncertainty for the weighing machine is calculated at the full-applied load. During the calibration, when the 60 kg standard weight applied on the weighing machine it indicated 59.99 kg.
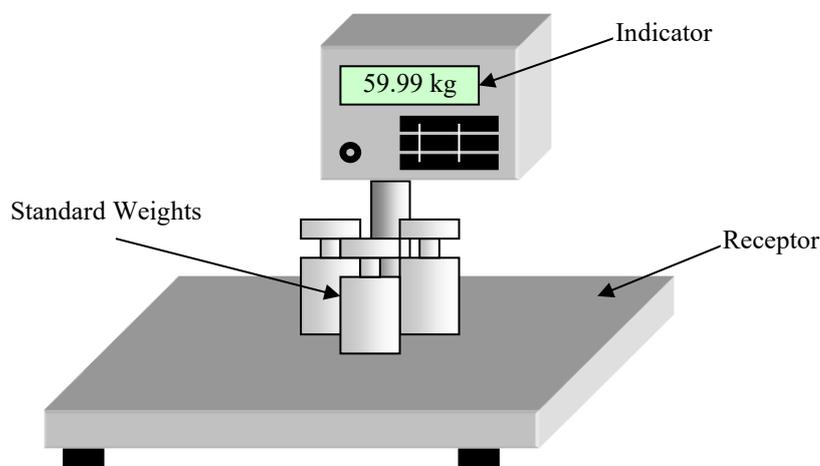


Figure 1: Weighing Machine being calibrated with 60 kg
(built up by three 20 kg) standard weights

## 2.0 MATHEMATICAL MODEL

The mathematical model for direct comparison method can be expressed as:

$$S = M + E \qquad \text{...(1)}$$

where
$S$ : Scale reading (the reading of the unit under test).
$M$ : Mass of the standard weight used to calibrate the electronic weighing machine.
$E$ : Error of the scale reading.

## EXAMPLE 11: CALIBRATION OF WEIGHING MACHINE

## 3.0     LAW OF PROPAGATION OF UNCERTAINTY

Where all input quantities are considered uncorrelated or called independent, the combined standard uncertainty $u_c$ (S) can be expressed by law of propagation of the scale reading (given in equation ( 1 )) as follows:

$$u_c^2(S) = \left(\frac{\partial S}{\partial M}\right)^2 u^2(M) + \left(\frac{\partial S}{\partial E}\right)^2 u^2(E)$$

where each corresponding sensitivity coefficient is as follows

$$c_M = \frac{\partial S}{\partial M}$$
$$= 1$$

$$c_E = \frac{\partial S}{\partial E}$$
$$= 1$$

$u(M)$   : standard uncertainty of the standard weight.

$u(E)$    : standard uncertainty due to the error of the scale reading.

and

$$u(M) = \sqrt{u^2(M_c) + u^2(M_d)}$$

where

$u(M_c)$ : standard uncertainty of the standard weight taken from its calibration report.

$u(M_d)$ : standard uncertainty of the standard weight drift within its calibration intervals.

and

$$u(E) = \sqrt{u^2(E_{res}) + u^2(E_b) + u^2(E_{rep})}$$

where

$u(E_{res})$ : standard uncertainty of the error of the scale reading due to its resolution.

$u(E_b)$  : standard uncertainty of the error of the scale reading due to air buoyancy effect.

$u(E_{rep})$ : standard uncertainty of the error of the scale reading due to its repeatability.

Therefore,

$$u_c^2(S) = c_M^2 \times u^2(M_c) + c_M^2 \times u^2(M_d) + c_E^2 \times u^2(E_{res}) + c_M^2 \times u^2(E_b) + c_E^2 \times u^2(E_{rep}) \quad ...(2)$$

# EXAMPLE 11: CALIBRATION OF WEIGHING MACHINE

## 4.0    EVALUATION OF UNCERTAINTY COMPONENTS

### *Type B*

4.1    $u_{M_c}$ , a set of standard weights was used to calibrate the weighing machine from 0 to 60 kg, the largest uncertainty would be at the largest built-up standard weight of 60 kg with the uncertainty of ± 1.8 g, at the given level of confidence of 95 %. Assuming a normal distribution, the degrees of freedom is ∞. Hence, the standard uncertainty,

$$u_{M_c} = \pm \left( \frac{1.8}{2} \right)$$
$$= \pm 0.9 \ \text{g}$$

4.2    $u_{M_d}$ , the estimated limits of the standard weights drift within the calibration interval is ± 0.3 g. It is assumed to be a rectangular distribution, hence the degrees of freedom is ∞. Thus, the standard uncertainty,

$$u_{M_d} = \pm \left( \frac{0.3}{\sqrt{3}} \right)$$
$$= \pm 0.173 \ \text{g}$$

4.3    $u_{E_{res}}$ , resolutions of the unit under test is 0.01 kg considered as a rectangular distribution, hence the degrees of freedom is ∞. Therefore, the standard uncertainty,

$$u_{E_{res}} = \pm \left( \frac{\frac{10}{2}}{\sqrt{3}} \right) \ \text{g}$$
$$= \pm \left( \frac{5}{\sqrt{3}} \right) \ \text{g}$$
$$= \pm 2.9 \ \text{g}$$

4.4    $u_{E_b}$ , the calibration was carried out in the controlled environment laboratory, the standard uncertainty of the weighing machine reading due to air buoyancy effect is estimated to be ± 0.55 g, assuming a rectangular distribution with degrees of freedom equals ∞. The standard uncertainty,

$$u_{E_b} = \pm \left( \frac{0.55}{\sqrt{3}} \right) \ \text{g}$$
$$= \pm 0.318 \ \text{g}$$

# EXAMPLE 11: CALIBRATION OF WEIGHING MACHINE

### *Type A*

4.5   $u_{E_{rep}}$ , standard uncertainty during the calibration were obtained from three intervals of the scale. At near zero, the standard deviation, $\sigma_0 = \pm\,0.01$ kg with the degrees of freedom, $v_0 = 9$. At half full load, the standard deviation, $\sigma_h = \pm\,0.01$ kg with the degrees of freedom, $v_h = 9$. At full load, the standard deviation, $\sigma_f = \pm\,0.01$ kg with the degrees of freedom, $v_f = 9$.

The pooled standard deviation,

$$\sigma_{E_{rep}} = \pm\sqrt{\frac{v_0 \times \sigma_0^2 + v_h \times \sigma_h^2 + v_f \times \sigma_f^2}{v_0 + v_h + v_f}}$$

$$= \pm\sqrt{\frac{9 \times 0.01^2 + 9 \times 0.01^2 + 9 \times 0.01^2}{9 + 9 + 9}}$$

$$= \pm 10 \ g$$

with the degrees of freedom $v_{E_{rep}} = 27$. Since the scale reading during calibration was measured one time at each calibrated point, hence, the standard uncertainty,

$$u_{E_{rep}} = \pm\left(\frac{\sigma_{E_{rep}}}{\sqrt{1}}\right)$$

$$= \pm 10 \ g$$

## 5.0   UNCERTAINTY BUDGET TABLE

| Source of Uncertainty | Symbol $x_i$ | Type | Uncertainty value | Probability distribution | Coverage factor | Standard uncertainty $u(x_i)$ | Sensitivity coefficient $c_i$ | $|c_i| \times u(x_i)$ | Degrees of freedom $v$ |
|---|---|---|---|---|---|---|---|---|---|
| Standard weight from calibration | $M_c$ | B | 1.8 g | Normal | 2 | 0.9 g | 1 | 0.9 g | $\infty$ |
| Standard weight drift | $M_d$ | B | 0.3 g | Rectangular | $\sqrt{3}$ | 0.173 g | 1 | 0.173 g | $\infty$ |
| Resolution of scale reading | $E_{res}$ | B | 5 g | Rectangular | $\sqrt{3}$ | 2.9 g | 1 | 2.9 g | $\infty$ |
| Air buoyancy effect on scale reading | $E_b$ | B | 0.55 g | Rectangular | $\sqrt{3}$ | 0.318 g | 1 | 0.318 g | $\infty$ |
| Repeatability of scale reading | $E_{rep}$ | A | 10 g | - | - | 10 g | 1 | 10 g | 27 |

**EXAMPLE 11: CALIBRATION OF WEIGHING MACHINE**

## 6.0 COMBINED STANDARD UNCERTAINTY

The Combined Standard Uncertainty $u_c(S)$ using equation (2)

$$u_c^2(S) = c_M^2 \times u^2(M_c) + c_M^2 \times u^2(M_d) + c_E^2 \times u^2(E_{res}) + c_M^2 \times u^2(E_b) + c_E^2 \times u^2(E_{rep})$$
$$= [c_M \times u(M_c)]^2 + [c_M \times u(M_d)]^2 + [c_E \times u(E_{res})]^2 + [c_E \times u(E_b)]^2 + [c_E \times u(E_{rep})]^2$$
$$= 0.9^2 + 0.173^2 + 2.9^2 + 0.318^2 + 10^2$$
$$u_c(S) = \pm 10.454 \text{ g}$$

## 7.0 EFFECTIVE DEGREES OF FREEDOM

Using Welch-Satterthwaite formula to calculate the effective degrees of freedom for combined standard uncertainty

$$\nu_{eff} = \frac{u_c^4(S)}{\dfrac{[c_M \times u(M_c)]^4}{\nu_{M_c}} + \dfrac{[c_M \times u(M_d)]^4}{\nu_{M_d}} + \dfrac{[c_E \times u(E_{Res})]^4}{\nu_{E_{Res}}} + \dfrac{[c_E \times u(E_b)]^4}{\nu_{E_b}} + \dfrac{[c_E \times u(E_{rep})]^4}{\nu_{E_{rep}}}}$$

$$= \frac{10.452^4}{\dfrac{0.9^4}{\infty} + \dfrac{0.173^4}{\infty} + \dfrac{2.9^4}{\infty} + \dfrac{0.006^4}{\infty} + \dfrac{10^4}{27}}$$

$$\approx 32$$

where

$\nu_{eff}$     : effective degrees of freedom of the combined standard uncertainty.

$\nu_{M_c}$     : degrees of freedom of the standard uncertainty of standard weight.

$\nu_{M_d}$     : degrees of freedom of the standard uncertainty due to standard weight drift.

$\nu_{E_{res}}$     : degrees of freedom of the standard uncertainty due to scale reading resolution.

$\nu_{E_b}$     : degrees of freedom of the standard uncertainty due to air buoyancy on scale reading.

$\nu_{E_{rep}}$     : degrees of freedom of the standard uncertainty due to scale reading repeatability.

**EXAMPLE 11: CALIBRATION OF WEIGHING MACHINE**

**8.0      EXPANDED UNCERTAINTY**

For $v_{eff} = 32$, $k = 2$ can be used at approximately 95 % level of confidence.  Therefore, the expanded uncertainty $U(S)$ is then calculated as follows:

$$U(S) = \pm\, k\, u_c(S)$$
$$= \pm\, 2 \times 10.451 \text{ g}$$
$$= \pm\, 20.902 \text{ g}$$
$$\approx \pm 0.02 \text{ kg}$$

**9.0      REPORTING OF RESULTS**

During the calibration, when the 60 kg standard weight applied on the weighing machine it indicated 59.99 kg.

The weighing machine indicated value of 59.99 kg having an expanded uncertainty of $\pm\, 0.02$ kg with a coverage factor of $k = 2$, assuming a normal distribution at a level of confidence of approximately 95 %.

# EXAMPLE 12: COORDINATE MEASURING MACHINE MEASUREMENT

## 1.0    INTRODUCTION

Using a coordinate measuring machine (CMM) to measure the length of a cylindrical rod from the centre of one end face to the other.
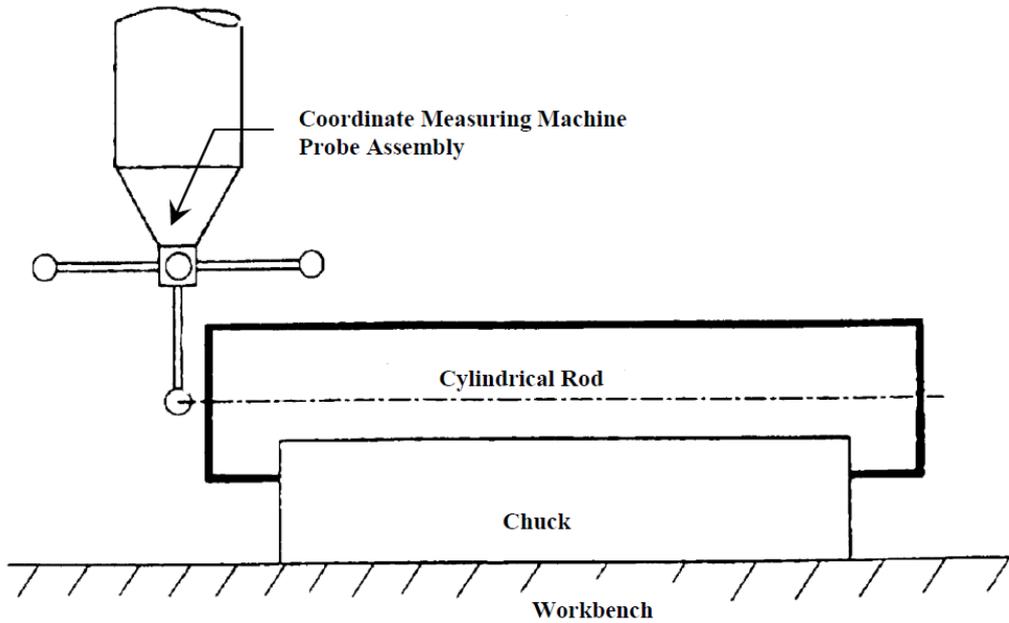


Figure 1: Measurement set-up

## 2.0    MATHEMATICAL MODEL

$$L_r = L_m \left( 1 - \alpha_m \Delta T_m + \alpha_r \Delta T_r \right) \quad ...(7)$$

where
$L_r$     : length of cylindrical rod at 20 °C.

$L_m$     : length as measured by CMM.

$\alpha_r$     : coefficient of thermal expansion of cylindrical rod.

$\alpha_m$     : coefficient of thermal expansion of CMM scale.

$\Delta T_r$     : temperature deviation of the cylindrical rod from 20 °C during measurement process.

$\Delta T_m$     : temperature deviation of CMM scale from 20 °C during measurement process.

**EXAMPLE 12: COORDINATE MEASURING MACHINE MEASUREMENT**

Contributory variances:

$$u_c^2(L_r) = \left(\frac{\partial L_r}{\partial L_m}\right)^2 u^2(L_m) + \left(\frac{\partial L_r}{\partial \alpha_m}\right)^2 u^2(\alpha_m) + \left(\frac{\partial L_r}{\partial \Delta T_m}\right)^2 u^2(\Delta T_m) +$$
$$\left(\frac{\partial L_r}{\partial \alpha_r}\right)^2 u^2(\alpha_r) + \left(\frac{\partial L_r}{\partial \Delta T_r}\right)^2 u^2(\Delta T_r) \qquad \ldots(2)$$

Sensitivity Coefficients are as follows:

$$\frac{\partial L_r}{\partial L_m} = 1 - \alpha_m \Delta T_m + \alpha_r \Delta T_r$$

$$\frac{\partial L_r}{\partial \alpha_m} = -L_m \Delta T_m$$

$$\frac{\partial L_r}{\partial \Delta T_m} = -L_m \alpha_m$$

$$\frac{\partial L_r}{\partial \alpha_r} = L_m \Delta T_r$$

$$\frac{\partial L_r}{\partial \Delta T_r} = L_m \alpha_r$$

Therefore,

$$u_c^2(L_r) = \left(1 - \alpha_m \Delta T_m + \alpha_r \Delta T_r\right)^2 u^2(L_m) + \left(-L_m \Delta T_m\right)^2 u^2(\alpha_m)$$
$$+ \left(-L_m \alpha_m\right)^2 u^2(\Delta T_m) + \left(L_m \Delta T_r\right)^2 u^2(\alpha_r) + \left(L_m \alpha_r\right)^2 u^2(\Delta T_r)$$

## 3.0 STANDARD UNCERTAINTY EVALUATION

3.1 Uncertainty of measured length, $u(L_m)$

From the ten repeated measurements taken, the mean measured result was 300.0008 mm with a standard deviation of 0.0020 mm. Therefore, the standard uncertainty, $u(L_{m1})$, equals the standard deviation of the mean of the ten measurements, 0.0006 mm, with degrees of freedom, $\nu(L_{m1}) = 10 - 1 = 9$.

**EXAMPLE 12: COORDINATE MEASURING MACHINE MEASUREMENT**

From the performance verification report on the CMM, the measurement uncertainty of the machine at a confidence level of approximately 95 % is $\pm$ (1.2 + 3$l$) µm, where $l$ is in metre. Therefore, the standard uncertainty, $u(L_{m2}) = (1.2 + 3 \times 0.3) / 2 = 1.05$ µm or 0.00105 mm with degrees of freedom, $v(L_{m2}) = \infty$, normal distribution is assumed.

Therefore,

$$u^2(L_m) = u^2(L_{m1}) + u^2(L_{m2})$$
$$= 0.0006^2 + 0.00105^2$$
$$u(L_m) = 0.001209 \ mm$$

with degrees of freedom, $v(L_m)$

$$v(L_m) = \frac{u^4(L_m)}{\dfrac{u^4(L_{m1})}{v(L_{m1})} + \dfrac{u^4(L_{m2})}{v(L_{m2})}}$$

$$= \frac{0.001209^4}{\dfrac{0.0006^4}{9} + \dfrac{0.00105^4}{\infty}}$$

$$= 148$$

3.2    Uncertainty of coefficient of thermal expansion of CMM scale, $u(\alpha_m)$

The coefficient of thermal expansion of CMM scale is given as $\alpha_m = 5 \times 10^{-6}$ / °C with an uncertainty represented by a rectangular distribution with boundaries $\pm 2 \times 10^{-6}$ / °C. Therefore, the standard uncertainty,

$$u(\alpha_m) = \frac{(2 \times 10^{-6} / °C)}{\sqrt{3}}$$
$$= 1.2 \times 10^{-6} / °C \qquad \text{with degrees of freedom } v(\alpha_m) = \infty$$

3.3    Uncertainty of temperature deviation of CMM scale from 20 °C, $u(\Delta T_m)$.

The temperature of CMM scale was measured as 19.9 °C with an uncertainty represented by a rectangular distribution with boundaries $\pm$ 0.5 °C. The standard uncertainty,

$$u(\Delta T_m) = \frac{0.5}{\sqrt{3}}$$
$$= 0.289 \ °C \qquad \text{with degrees of freedom } v(\Delta T_m) = \infty$$

# EXAMPLE 12: COORDINATE MEASURING MACHINE MEASUREMENT

## 4.0 SENSITIVITY COEFFICIENTS

$$\frac{\partial L_r}{\partial L_m} = 1 - \alpha_m \Delta T_m + \alpha_r \Delta T_r$$
$$= 1 - (5 \times 10^{-6})(0.1) + (11.5 \times 10^{-6})(0.1)$$
$$= 1.00000065$$

$$\frac{\partial L_r}{\partial \alpha_m} = -L_m \Delta T_m$$
$$= (300.0008)(0.1)$$
$$= 30.00008 \ mm \ ℃$$

$$\frac{\partial L_r}{\partial \Delta T_m} = -L_m \alpha_m$$
$$= -(300.0008)(5 \times 10^{-6})$$
$$= -0.0015 \ mm/℃$$

$$\frac{\partial L_r}{\partial \alpha_r} = L_m \Delta T_r$$
$$= (-300.0008)(0.1)$$
$$= -30.00008 \ mm \ ℃$$

$$\frac{\partial L_r}{\partial \Delta T_r} = L_m \alpha_r$$
$$= (300.0008)(11.5 \times 10^{-6})$$
$$= 0.00345 \ mm/℃$$

## 5.0 UNCERTAINTY BUDGET TABLE

| Source of Uncertainty | Symbol $x_i$ | Type | Uncertainty value | Probability distribution | Coverage factor | Standard uncertainty $u(x_i)$ | Sensitivity coefficient $c_i$ | $\lvert c_i \rvert$ x $u(x_i)$ mm | Degrees of freedom $\nu$ |
|---|---|---|---|---|---|---|---|---|---|
| Repeatability of reading | $u(L_{m1})$ | A | - | - | - | 0.6 µm | 1.00000065 | 0.0006 | 9 |
| CMM specification | $u(L_{m2})$ | B | $\pm (1.2 + 3\ l)$ µm | Normal | 2 | 1.05 µm | 1.00000065 | 0.00105 | ∞ |
| Coefficient of thermal expansion of CMM scale | $u(\alpha_m)$ | B | $\pm 2 \times 10^{-6}$/ º C | Rectangular | $\sqrt{3}$ | $1.2 \times 10^{-6}$ / º C | - 30.000 08 mm ºC | $3.6 \times 10^{-5}$ | ∞ |
| Temperature deviation of CMM scale | $u(\Delta T_m)$ | B | $\pm 0.5$ º C | Rectangular | $\sqrt{3}$ | 0.289 º C | - 0.0015 mm / ºC | 0.0004 | ∞ |
| Coefficient of thermal expansion of rod | $u(\alpha_r)$ | B | $\pm 2 \times 10^{-6}$/ º C | Rectangular | $\sqrt{3}$ | $1.2 \times 10^{-6}$ / º C | 30.000 08 mm ºC | $3.6 \times 10^{-5}$ | ∞ |
| Temperature deviation of rod | $u(\Delta T_r)$ | B | $\pm 0.5$ º C | Rectangular | $\sqrt{3}$ | 0.289 º C | 0.00345 mm / ºC | 0.0010 | ∞ |

## EXAMPLE 12: COORDINATE MEASURING MACHINE MEASUREMENT

### 6.0 COMBINED STANDARD UNCERTAINTY, $u_c(L_r)$

$$
\begin{aligned}
u_c^2(L_r) &= (1 - \alpha_m \Delta T_m + \alpha_r \Delta T_r)^2 u^2(L_m) + (-L_m \Delta T_m)^2 u^2(\alpha_m) + (-L_m \alpha_m)^2 u^2(\Delta T_m) \\
&\quad + (L_m \Delta T_r)^2 u^2(\alpha_r) + (L_m \alpha_r)^2 u^2(\Delta T_r) \\
&= 1.00000065^2 \, x \, 0.001209^2 + (-30.00008)^2 \, x \, (1.2 \times 10^{-6})^2 + (-0.0015)^2 \, x \, 0.289^2 \\
&\quad + 30.00008^2 \, x \, (1.2 \, x 10^{-6})^2 + 0.00345^2 \, x \, 0.289^2 \\
&= 0.000002646 \ mm^2
\end{aligned}
$$

$$
u_c(L_r) = 0.0016267 \ mm
$$

with effective degrees of freedom,

$$
\nu_{eff} = \frac{u_c^4(L_r)}{\dfrac{\left[(1 - \alpha_m \Delta T_m + \alpha_r \Delta T_r)u(L_m)\right]^4}{\nu(L_m)} + \dfrac{\left[(-L_m \Delta T_m)u(\alpha_m)\right]^4}{\nu(\alpha_m)} + \dfrac{\left[(-L_m \alpha_m)u(\Delta T_m)\right]^4}{\nu(\Delta T_m)} + \dfrac{\left[(L_m \Delta T_r)u(\alpha_r)\right]^4}{\nu(\alpha_r)} + \dfrac{\left[(L_m \alpha_r)u(\Delta T_r)\right]^4}{\nu(\Delta T_r)}}
$$

$$
= \frac{0.00161^4}{\dfrac{\left[(1 - 5 \, x 10^{-6} \, x \, 0.1 + 11.5 \, x 10^{-6} \, x \, 0.1)x \, 0.001209\right]^4}{148} + 0 + 0 + 0 + 0}
$$

$$
= 485
$$

### 7.0 EXPANDED UNCERTAINTY

For $\nu_{eff}(L_r) = 485$, $k = 2$ can be used at an approximately 95 % level of confidence level.
Therefore,

$$
\begin{aligned}
U &= k \, u_c \\
&= 2 \times 0.0016267 \\
&= 0.0032 \ mm
\end{aligned}
$$

### 8.0 MEAN MEASURED RESULT

$$
\begin{aligned}
L_r &= L_m(1 - \alpha_m \Delta T_m + \alpha_r \Delta T_r) \\
&= 300.0008 \times (1 - (5 \times 10^{-6})(0.1) + (11.5 \times 10^{-6})(0.1) \,) \\
&= 300.0010 \ mm
\end{aligned}
$$

### 9.0 REPORTING OF RESULTS

The measured length of the cylindrical rod was found to be 300.0010 mm. The associated expanded uncertainty of measurement is 0.0032 mm, estimated at a level of confidence of approximately 95 % with coverage factor $k = 2$.

**EXAMPLE 13: MICROMETER CALIBRATION**

## 1.0 INTRODUCTION

The calibrated instrument is a digital micrometer of 0-25 mm range with resolution of 0.001 mm.
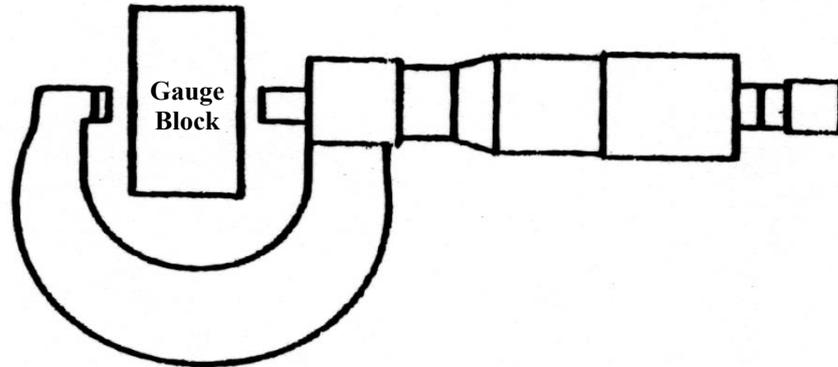


Figure 1: Measurement set-up

The error of measurement is obtained by comparison against gauge blocks of the following sizes:

3.1, 6.5, 9.7, 12.5, 15.8, 19, 21.9, 25 mm

## 2.0 MATHEMATICAL MODEL

$$E_{20} = IR_{20} - G_{20} \qquad \qquad ...(1)$$

where

$E_{20}$ : error of measurement obtained at 20 $^\circ$C.

$IR_{20}$ : instrument reading at 20 $^\circ$C.

$G_{20}$ : gauge block value given in the calibration report at 20 $^\circ$C.

With consideration of the temperature effect on the instrument under calibration, the above expression can be re-written as:

$$E_{20} = IR_t \, (1 - \alpha_{IR} \, \Delta t_{IR}) - G_{20} \qquad \qquad ...(2)$$

where

$IR_t$ : instrument reading at temperature $t$.

$\alpha_{IR}$ : coefficient of thermal expansion of the instrument.

$\Delta t_{IR}$ : temperature deviation of instrument from 20 $^\circ$C, ie $(t-20)$.

**EXAMPLE 13: MICROMETER CALIBRATION**

Contributory variances:

$$u_c^2(E_{20}) = \left(\frac{\partial E_{20}}{\partial IR_t}\right)^2 u^2(IR_t) + \left(\frac{\partial E_{20}}{\partial \alpha_{IR}}\right)^2 u^2(\alpha_{IR}) + \left(\frac{\partial E_{20}}{\partial \Delta t_{IR}}\right)^2 u^2(\Delta t_{IR}) + \left(\frac{\partial E_{20}}{\partial G_{20}}\right)^2 u^2(G_{20}) \qquad ...(3)$$

Sensitivity coefficients:

$$\left(\frac{\partial E_{20}}{\partial IR_t}\right) = 1 - \alpha_{IR}\Delta t_{IR}$$

$$\left(\frac{\partial E_{20}}{\partial \alpha_{IR}}\right) = -IR_t\Delta t_{IR}$$

$$\left(\frac{\partial E_{20}}{\partial \Delta t_{IR}}\right) = -IR_t\alpha_{IR}$$

$$\left(\frac{\partial E_{20}}{\partial G_{20}}\right) = -1$$

Therefore,

$$u_c^2(E_{20}) = (1 - \alpha_{IR}\Delta t_{IR})^2\, u^2(IR_t) + (IR_t\Delta t_{IR})^2 u^2(\alpha_{IR}) + (IR_t\alpha_{IR})^2 u^2(\Delta t_{IR}) + u^2(G_{20})$$

## 3.0 MEASURED RESULTS

The results tabulated below are based on the assumptions that measurements were taken at temperature of 20.5 $^\circ$C and the instrument has a coefficient of thermal expansion of $11\times10^{-6}$ / $^\circ$C.

| Nominal Length (mm) | Gauge Block Value, $G_{20}$ (mm) | Instrument Reading, $IR_t$ (mm) | | | | Error, $E_{20}$ (mm) | Standard Deviation (μm) |
|---|---|---|---|---|---|---|---|
| | | 1st Reading | 2nd Reading | 3rd Reading | Mean | | |
| 0.0 | - | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 3.1 | 3.100 05 | 3.101 | 3.100 | 3.101 | 3.1007 | + 0.001 | 0.577 |
| 6.5 | 6.499 97 | 6.501 | 6.501 | 6.500 | 6.5007 | + 0.001 | 0.577 |
| 9.7 | 9.700 00 | 9.699 | 9.700 | 9.699 | 9.6993 | - 0.001 | 0.577 |
| 12.5 | 12.500 04 | 12.501 | 12.501 | 12.501 | 12.5010 | + 0.001 | 0.000 |
| 15.8 | 15.800 03 | 15.800 | 15.801 | 15.800 | 15.8003 | 0.000 | 0.577 |
| 19 | 19.000 07 | 19.001 | 19.001 | 19.001 | 19.0010 | + 0.001 | 0.000 |
| 21.9 | 21.900 00 | 21.900 | 21.900 | 21.900 | 21.9000 | 0.000 | 0.000 |
| 25 | 25.000 07 | 25.001 | 25.001 | 25.000 | 25.0007 | 0.000 | 0.577 |

### EXAMPLE 13: MICROMETER CALIBRATION

## 4.0    STANDARD UNCERTAINTY EVALUATION

4.1    Uncertainty of instrument reading, $u(IR_t)$

Generally, the highest standard deviation is used to estimate the standard uncertainty due to repeated measurements, $u(Rpt)$. From the three measurements taken, we obtain:

$$u(Rpt) = \frac{0.577}{\sqrt{n}} \qquad \text{where } n = 3$$

$$= \frac{0.577}{\sqrt{3}}$$

$$= 0.333 \; \mu m \quad \text{with degrees of freedom } \nu(Rpt) = n - 1 = 2$$

The resolution (0.001 mm or 1 μm) of the micrometer is considered to be full-width of rectangular distribution and its associated standard uncertainty, $u(Res)$ is given as:

$$u(Res) = \frac{0.001}{\sqrt{12}}$$

$$= 0.289 \; \mu m \quad \text{with degrees of freedom } \nu(Res) = \infty$$

Therefore,

$$u^2(IR_t) = u^2(Rpt) + u^2(Res)$$

$$= 0.333^2 + 0.289^2$$

$$u(IR_t) = 0.441 \; \mu m$$

$$\nu(IR_t) = \frac{u^4(IR_t)}{\dfrac{u^4(Rpt)}{\nu(Rpt)} + \dfrac{u^4(Res)}{\nu(Res)}}$$

$$= \frac{0.441^4}{\dfrac{0.333^4}{2} + \dfrac{0.289^4}{\infty}}$$

$$= 6.2$$

$$\approx 6$$

4.2    Uncertainty of gauge block, $u(G_{20})$

Assuming that the uncertainty given in the calibration report with coverage factor $k = 2$ for gauge block size up to 25 mm is 0.10 μm, we obtain:

$$u(G_{20}) = \frac{0.10}{2}$$

$$= 0.05 \; \mu m \quad \text{with degrees of freedom } \nu(G_{20}) = \infty$$

**EXAMPLE 13: MICROMETER CALIBRATION**

4.3 Uncertainty of coefficient of thermal expansion of the instrument, $u(\alpha_{IR})$

Given that the uncertainty of the coefficient of thermal expansion is represented by a rectangular distribution with boundaries of $\pm\, 2 \times 10^{-6} / °C$, we obtain:

$$u(\alpha_{IR}) = \frac{2 \times 10^{-6} / °C}{\sqrt{3}}$$

$$= 1.2 \times 10^{-6} / °C \qquad \text{with degrees of freedom } v(\alpha_{IR}) = \infty$$

4.4 Uncertainty of temperature deviation of instrument from 20 °C, $u(\Delta t_{IR})$

Given that the temperature was measured as 20.5 °C with an uncertainty represented by a rectangular distribution with boundaries of $\pm\, 0.5$ °C, we obtain:

$$u(\Delta t_{IR}) = \frac{0.5}{\sqrt{3}}$$

$$= 0.289 \; °C \qquad \text{with degrees of freedom } v(\Delta t_{IR}) = \infty$$

**5.0 SENSITIVITY COEFFICIENTS**

$$\frac{\partial E_{20}}{\partial IR_t} = 1 - \alpha_{IR} \Delta t_{IR}$$

$$= 1 - 11 \times 10^{-6} \times (20.5 - 20)$$

$$= 0.99999$$

Taken into consideration the maximum size of measurement,

$$\frac{\partial E_{20}}{\partial \alpha_{IR}} = -IR_t \Delta t_{IR}$$

$$= -25.0007 \times (20.5 - 20)$$

$$= -12.5004 \; \text{mm}°C$$

$$\frac{\partial E_{20}}{\partial \Delta t_{IR}} = -IR_t \alpha_{IR}$$

$$= -25.0007 \times 11 \times 10^{-6}$$

$$= -0.00028 \; \text{mm}/°C$$

$$\frac{\partial E_{20}}{\partial G_{20}} = -1$$

## EXAMPLE 13: MICROMETER CALIBRATION

## 6.0 UNCERTAINTY BUDGET TABLE

| Source of Uncertainty | Symbol $x_i$ | Type | Uncertainty value | Probability distribution | Coverage factor | Standard uncertainty $u(x_i)$ | Sensitivity coefficient $c_i$ | $\lvert c_i \rvert$ x $u(x_i)$ mm | Degrees of freedom $\nu$ |
|---|---|---|---|---|---|---|---|---|---|
| Repeatability of reading | $u\,(Rpt)$ | A | - | - | - | 0.333 µm | 1 | 0.000 333 | 2 |
| Resolution of instrument | $u\,(Res)$ | B | 0.001 mm | Rectangular | $\sqrt{12}$ | 0.289 µm | 1 | 0.000 289 | ∞ |
| Gauge block | $u\,(G_{20})$ | B | 0.10 µm | Normal | 2 | 0.05 µm | - 1 | 0.000 05 | ∞ |
| Coefficient of thermal expansion of instrument | $u\,(\alpha_{IR})$ | B | $\pm\, 2 \times 10^{-6}$ / °C | Rectangular | $\sqrt{3}$ | $1.2 \times 10^{-6}$ /°C | - 12.5004 mm °C | $1.5 \times 10^{-5}$ | ∞ |
| Temperature deviation of instrument | $u\,(\Delta t_{IR})$ | B | $\pm\, 0.5$ °C | Rectangular | $\sqrt{3}$ | 0.289 °C | - 0.000 28 mm / °C | $8.1 \times 10^{-5}$ | ∞ |

## 7.0 COMBINED STANDARD UNCERTAINTY, $u(E_{20})$

$$u_c^2(E_{20}) = (1 - \alpha_{IR}\Delta t_{IR})^2 u^2(IR_t) + (IR_t \Delta t_{IR})^2 u^2(\alpha_{IR}) + (IR_t \alpha_{IR})^2 u^2(\Delta t_{IR}) + u^2\,(G_{20})$$
$$= (0.99999)^2 (0.000441)^2 + (12.5004)^2 \times (1.2 \times 10^{-6})^2 + (0.00028)^2 \times (0.289)^2 + (0.00005)^2$$
$$= 2.0375 \times 10^{-7}$$
$$u_c(E_{20}) = 0.451 \; um$$

Effective degrees of freedom is thus:

$$v_{eff}(E_{20}) = \frac{u_c^4(E_{20})}{\dfrac{(1 - \alpha_{IR}\Delta t_{IR})^4 u^4(IR_t)}{v(IR_t)} + \dfrac{(IR_t \Delta t_{IR})^4 u^4(\alpha_{IR})}{v(\alpha_{IR})} + \dfrac{(IR_t \alpha_{IR})^4 u^4(\Delta t_{IR})}{v(\Delta t_{IR})} + \dfrac{u^4(G_{20})}{v(G_{20})}}$$
$$= \frac{0.451^4}{\dfrac{0.99999^4 \times 0.441^4}{6} + 0 + 0 + 0}$$
$$= 6.56$$
$$\approx 6$$

## 8.0 EXPANDED UNCERTAINTY

From t-distribution table, at $v_{eff}(E_{20}) = 6$

and at an approximately 95 % level of confidence, $k = 2.52$.

Therefore,

$$U = 2.52 \text{ x } 0.451$$
$$= 1.137 \; \text{µm}$$

**EXAMPLE 13: MICROMETER CALIBRATION**

**9.0    UNCERTAINTY OF THE RANGE OF ERROR OF TRANSVERSE OF MICROMETER SCREW**

The range of error of transverse of micrometer screw is given as:

$$(E_{20})_{max} - (E_{20})_{min} = 0.002 \text{ mm}$$

Hence,

$$u(range\ of\ error) = \sqrt{2} \times u_c(E_{20})$$
$$= 0.638\ um$$

with degrees of freedom,

$$v(range\ of\ error) = \frac{0.638^4}{\frac{0.451^4}{6} + \frac{0.451^4}{6}}$$
$$= 12.02$$

From t-distribution table, $\nu$ (*range of error*) = 12 and at an approximately 95 % level of confidence, $k$ = 2.23. Therefore,

$$U = 2.23 \text{ x } 0.638$$
$$= 1.423 \text{ μm}$$

**10.0    REPORTING OF RESULTS**

The range of error of transverse of micrometer screw was found to be 0.002 mm. The associated expanded uncertainty of measurement is 1.423 μm, estimated at a level of confidence of approximately 95 % with coverage factor $k$ = 2.23.

## EXAMPLE 14: ILLUMINATION METER CALIBRATION

### 1.0    INTRODUCTION

A schematic diagram of the calibration set-up is shown in Fig.1. A tungsten lamp of (calibrated) luminous intensity value is used as the standard, and the test meter is positioned at a distance away from the lamp. Calibration is done by comparing the reading of the test meter with the illuminance value determined by the inverse-square law from the luminous intensity of the lamp.
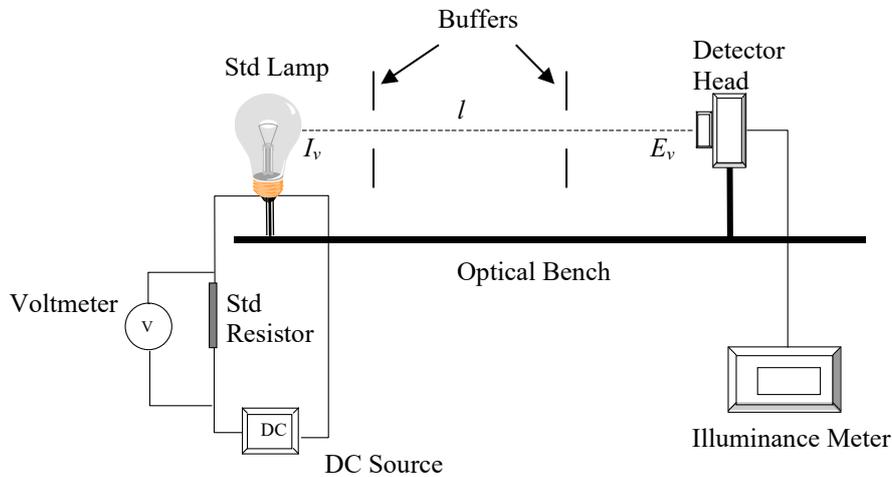


Figure 1: Schematic diagram of the calibration set-up

### 2.0    MATHEMATICAL MODEL

The luminous-intensity standard lamp generates an illuminance of known value at the front surface of the detector head of the test meter positioned in front of the lamp with a reading $E_v$ :

$$E_v = I_v \frac{\cos\theta}{l^2} + scattered\ light + \Delta E_v$$

where
$I_v$        : luminous intensity of the standard lamp run at a given current at which the correlated colour temperature of the lamp is approximately 2856K.

$l$         : distance between the lamp filament and the front surface of the test meter.

$I_v \frac{\cos\theta}{l^2}$ : the illuminance produced by the lamp according to the inverse-square law.

$\theta$        : the angle of deviation from normal incidence at the detector head.

$\Delta E_v$     : the error of the test meter; scattered light refers to input signal to the detector from light scattered or reflected from the wall, curtains or other surfaces in the optical set-up other than that from direct illumination by the lamp. Normally, the second and third terms are much smaller than the first one.

## EXAMPLE 14: ILLUMINATION METER CALIBRATION

For simplicity, we will assume in the following discussions that

a) the calibration is conducted in a dark room and sufficient measure is taken during calibration so that the contribution of scattered light is negligible;
b) both the lamp and the detector head of the test meter are well aligned to a common optical axis determined by the optical bench so that $cos\ \theta = 1$.

The measurement equation can then be simplified as

$$E_v = \frac{I_v}{l^2} + \Delta E_v \qquad \qquad ...(1)$$

For incandescent tungsten lamps, the luminous intensity varies with electrical current flowing through it according to

$$I_v = I_v(0)\left(\frac{i}{i_o}\right)^{6.25} \qquad \qquad ...(2)$$

where $I_v(0)$ is the reported (calibrated) luminous intensity value of the lamp under a current $i_0$. The current ($i$) of the lamp is determined by measuring the voltage drop ($V$) across a standard resistor (resistance value $R$ ) that is connected in series with the lamp:

$$i = \frac{V}{R}$$

and

$$R = R_{20}\left[1 + \alpha(T - 20) + \beta(T - 20)^2\right] \qquad \qquad ...(3)$$

where $R_{20}$ is the resistance of the standard resistor at 20 °C, $\alpha$ and β are the temperature coefficients of the resistor, and $T$ is the resistor temperature in Celsius.


## 3.0    UNCERTAINTY EQUATION

The main sources of measurement uncertainty can be modelled from eq (1) through partial differentiation:

$$\left(\frac{dE_v}{E_v}\right)^2 = \left(\frac{dI_v}{I_v}\right)^2 + \left[2\left(\frac{dl}{l}\right)\right]^2 + \left(\frac{d(\Delta E_v)}{E_v}\right)^2 \qquad \qquad ...(4)$$

where the last term represents the standard uncertainty of the error and the assumption $\Delta E_v \langle\langle \frac{I_v}{l^2}$ is used in the derivation.

## EXAMPLE 14: ILLUMINATION METER CALIBRATION

According to eq (2)

$$\left(\frac{dI_v}{I_v}\right)^2 = \left(6.25 \times \frac{di}{i}\right)^2 + \left(\frac{dI_v(0)}{I_v(0)}\right)^2 \qquad \text{...(5)}$$

The second term represents the standard uncertainty (type B) of the lamp calibration. The first term is the standard uncertainty of current measurement determined by

$$\left(\frac{di}{i}\right)^2 = \left(\frac{dV}{V}\right)^2 + \left(\frac{dR}{R}\right)^2 \qquad \text{...(6)}$$

and

$$\left(\frac{dV}{V}\right)^2 = \left(u(V_A)\right)^2 + \left(u(V_B)\right)^2 \qquad \text{...(7)}$$

$$\frac{dR}{R} = \frac{dR_{20}}{R_{20}} + \frac{[\alpha + 2\beta(T-20)] \times \partial T}{1 + \alpha(T-20) + \beta(T-20)^2}$$
$$\approx \frac{dR_{20}}{R_{20}} + [\alpha + 2\beta(T-20)] \times \partial T \qquad \text{...(8)}$$

where the first and second terms in eq (7) represent the random (type A) uncertainty due to measurement repeatability and systematic (type B) uncertainties due to calibration of the voltmeter respectively.

Substituting eqs (5) - (8) into eq (4), we have

$$\left(\frac{dE_v}{E_v}\right)^2 = \left(2\frac{dl}{l}\right)^2 + 6.25^2\left[\left(\frac{dV}{V}\right)^2 + \left(\frac{dR}{R}\right)^2\right] + \left(\frac{dI_v(0)}{I_v(0)}\right)^2 + \left(\frac{d(\Delta E_v)}{E_v}\right)^2$$

or

$$\left(\frac{dE_v}{E_v}\right)^2 = \left(2 \times u(l)\right)^2 + \left(6.25 \times u(V_A)\right)^2 + \left(6.25 \times u(V_B)\right)^2 + \left(6.25 \times u(R)\right)^2 + u(I_v)^2 + u(\Delta E_v)^2 \text{ ...(9)}$$

where
$u(l)$ : standard uncertainty of distance measurement.
$u(V_A)$ : type A standard uncertainty of voltage measurement (voltmeter repeatability).
$u(V_B)$ : type B standard uncertainty of voltage measurement due to voltmeter accuracy specification.
$u(R)$ : standard uncertainty of resistance value of the standard resistor.
$u(I_v)$ : standard uncertainty of standard lamp calibration.
$u(\Delta E_v)$ : standard uncertainty of the error of illuminance meter measurement.

## EXAMPLE 14: ILLUMINATION METER CALIBRATION

### 4.0    NUMERICAL EXAMPLE

4.1.    *u(l),* uncertainty of distance measurement

It can be determined using the accuracy of the scale on the bench and repeatability of the distance measurement. From past data, it is typically $\pm 1$ mm at a distance 1 m or $\pm 0.1$ % and can be considered as a type B uncertainty with a rectangular distribution. The relative standard uncertainty is therefore

$$u(l) = \frac{0.1\ \%}{\sqrt{3}}$$
$$= 0.0577\ \%$$

The degrees of freedom is $\infty$.

4.2.    *u(V_A)*, type A standard uncertainty of voltage measurement (voltmeter repeatability), is determined by the standard deviation of the test data shown in the following table:

| Data (mV) | 36.6209 |
|---|---|
| | 36.6210 |
| | 36.6209 |
| | 36.6209 |
| | 36.6210 |
| Mean (mV) | 36.62094 |
| STD (mV) | 0.000055 |
| STD of Mean (mV) | 0.000024 |
| Standard uncertainty (relative STD of Mean ) | 6.6E-5 % |
| Number of measurement (n) | 5 |
| Degrees of Freedom (n-1) | 4 |

Note: STD = standard deviation.

4.3.    *u(V_B),* type B standard uncertainty of voltage measurement due to voltmeter accuracy specification.

For the range of concern (100 mV), it is given by the manufacturer as $\pm$ (0.0015 % of reading + 0.0003 % of range) under the environmental condition 23 °C $\pm$ 1 °C within 24 hours. Using the data above, we have

$$u(V_B) = 0.0015\ \% \text{ x } 36.6 + 0.0003\ \% \text{ x } 100$$
$$= 0.00085\ \text{mV}$$
$$\Rightarrow 0.0023\ \%$$

with a degrees of freedom of $\infty$.

# EXAMPLE 14: ILLUMINATION METER CALIBRATION

4.4.    *u(R),* relative standard uncertainty of resistance value of the standard resistor, is determined by eq (8).

In this example, the values given by the manufacturer are $\alpha = 2.5 \times 10^{-6}$ °C$^{-1}$ and $\beta = 0.55 \times 10^{-6}$ °C$^{-1}$. The DC resistance of the standard resistor is given in it's calibration report as 0.009 999 26 $\Omega \pm 0.000\ 000\ 05\ \Omega$, temperature corrected to 23 °C, at a level of confidence of approximately 95 % with a coverage factor $k = 2$. This corresponds to a relative standard uncertainty $\pm 0.000\ 25$ %, applicable also to dR$_{20}$/R$_{20}$. Assuming the temperature variation ( dT ) of the resistor during calibration is $\pm$ 1 °C under typical calibration lab environmental conditions T = 23 $\pm$ 1 °C, we have

$$u(R) = \sqrt{\left( \frac{dR_{20}}{R_{20}} \right)^2 + \left( \alpha + 2\beta(T - 20) \right)^2 \partial T^2}$$

$$= 0.00063 \ \%$$

with a degrees of freedom of $\infty$.

4.5    *u(I$_v$)* - relative standard uncertainty of standard lamp calibration

An expanded uncertainty of $\pm$ 1.1% is given in the calibration report at a confidence level approximately 95 % with a coverage factor of 2. This corresponds to a relative standard uncertainty $\pm$ 0.55 % with a degrees of freedom of $\infty$.

4.6    *u($\Delta E_v$)* - standard uncertainty of the error of illuminance meter measurement (repeatability)

This is determined by measurement data given below

| | Applied value $(I_v(0)/l^2)$ | Measured value ( $E_v$ ) | Error ($\Delta E_v = E_v - I_v(0)/l^2$) |
|---|---|---|---|
| data (lux) | 1599.6 | 1559.2 | - 40.4 |
| | | 1560.2 | - 39.4 |
| | | 1560.2 | - 39.4 |
| | | 1559.2 | - 40.4 |
| | | 1559.2 | - 40.4 |
| Mean | | 1559.6 | - 40.0 |
| STD | | 0.548 | |
| STD of the mean: | | 0.245 | |
| Standard uncertainty (relative STD of the mean) | | 0.016 % | |
| Number of measurements | | 5 | |
| Degrees of freedom | | 4 | |

**EXAMPLE 14: ILLUMINATION METER CALIBRATION**

Notes: From the definition given in eq (9), $u(\Delta E_v) = d(\Delta E_v)/E_v$. Since the applied value in the calibration is a constant, the standard deviations of the error $d(\Delta E_v)$ and of the test meter reading $(dE_v)$ are identical. Therefore, $u(\Delta E_v)$ is equal to the relative standard deviation of the mean of the test meter readings.

**5.0    UNCERTAINTY BUDGET TABLE**

| Source of Uncertainty | Symbol $x_i$ | Type | Uncertainty value (%) | Probability distribution | Coverage factor | Standard uncertainty (%) $u(x_i)$ | Sensitivity coefficient $c_i$ | $|c_i|$ x $u(x_i)$ (%) | Degrees of freedom $v_i$ |
|---|---|---|---|---|---|---|---|---|---|
| Uncertainty of distance measurement | $u(l)$ | B | 0.1 | Rectangular | 1.73 | 0.058 | -2 | 0.12 | ∞ |
| Repeatability of voltage measurement | $u(V_A)$ | A | 0.000066 | - | - | 0.000066 | 6.25 | 0.00041 | 4 |
| Accuracy of Voltmeter | $u(V_B)$ | B | 0.0023 | Rectangular | 1.73 | 0.0013 | 6.25 | 0.0084 | ∞ |
| Uncertainty of std resistor | $u(R)$ | B | 0.00063 | Normal | 1 | 0.00063 | -6.25 | 0.0039 | ∞ |
| Uncertainty of std lamp | $u(I_v)$ | B | 1.1 | Normal | 2 | 0.55 | 1 | 0.55 | ∞ |
| Repeatability of test meter reading | $u(\Delta E_v)$ | A | 0.016 | - | - | 0.016 | 1 | 0.016 | 5 |

**6.0    COMBINED STANDARD UNCERTAINTY**

The combined relative standard uncertainty $u_c(E_v)$ is given by eq (9), which can be rewritten as below

$$u_c{}^2(E_v) = \left(\frac{dE_v}{E_v}\right)^2$$
$$= (2 \times u(l))^2 + (6.25 \times u(V_A)^2 + (6.25 \times u(V_B)^2 + (6.25 \times u(R))^2 + u(I_v)^2 + u(\Delta E_v)^2$$

Substituting the data from Section 4, we have

$$u_c(y) = \left[ \begin{array}{l} (2 \times 5.77 \times 10^{-2})^2 + (6.25 \times 6.6 \times 10^{-5})^2 + (6.25 \times 2.3 \times 10^{-3})^2 \\ + (6.25 \times 6.3 \times 10^{-4})^2 + (5.5 \times 10^{-1})^2 + (1.6 \times 10^{-2})^2 \end{array} \right]^{1/2} \%$$
$$= 0.56 \ \%$$

**EXAMPLE 14: ILLUMINATION METER CALIBRATION**

## 7.0 EFFECTIVE DEGREES OF FREEDOM

It can be calculated from the general formulae given in the Guide and is essentially infinity in this example.

$$
\begin{aligned}
v_{eff} &= \frac{u_c^4(E_v)}{\displaystyle\sum_{i=1}^{n} \frac{c_i^4 u^4(x_i)}{v_i}} \\[2mm]
&= u_c^4(y)\left[\frac{c_1^4 u^4(l)}{v_1} + \frac{c_2^4 u^4(V_A)}{v_2} + \frac{c_3^4 u^4(V_B)}{v_3} + \frac{c_4^4 u^4(R)}{v_4} + \frac{c_5^4 u^4(I_v)}{v_5} + \frac{c_6^4 u^4(\Delta E_v)}{v_6}\right]^{-1} \\[2mm]
&= 0.56^4\left[\frac{c_1^4 u^4(l)}{\infty} + \frac{\left(6.25 \times 6.6 \times 10^{-5}\right)^4}{4} + \frac{c_3^4 u^4(V_B)}{\infty} + \frac{c_4^4 u^4(R)}{\infty} + \frac{c_5^4 u^4(I_v)}{\infty} + \frac{\left(1.6 \times 10^{-2}\right)^4}{4}\right]^{-1} \\[2mm]
&= 6.0 \times 10^6 \\
&= \infty
\end{aligned}
$$

## 8.0 EXPANDED UNCERTAINTY

For effective degrees of freedom $v_{eff} = \infty$, $k = 2$ is used at approximately 95 % level of confidence. Therefore, the expanded uncertainty

$$
\begin{aligned}
U &= k\, u_c(E_v) \\
&= 2 \times 0.56\ \% \\
&= 1.1\ \%
\end{aligned}
$$

## 9.0 REPORTING OF RESULTS

| Description | Applied value (lux) | Indicated value (lux) | Deviation (%) (lux) |
|---|---|---|---|
| Range:<br>　　Auto | 1599.6 | 1559.6 | -40 (-2.3%) |

The expanded measurement uncertainty is ± 1.1 %, estimated at a level of confidence of approximately 95 % with a coverage factor $k = 2$.